Reflecting Diffusion Process on Time-Inhomogeneous Manifolds with Boundary

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Abstract

Let $L_t := \Delta_t + Z_t$ for a $C^{1,1}$-vector field Z on a differential manifold M with boundary ∂M, where Δ_t is the Laplacian induced by a time dependent metric g_t differentiable in $t \in [0, T_c)$. We first introduce the reflecting diffusion process generated by L_t and establish the derivative formula for the associated diffusion semigroup; then construct the couplings for the reflecting L_t-diffusion processes by parallel and reflecting displacement, which implies the gradient estimates of the associated heat semigroup; and finally, present a number of equivalent inequalities for the curvature lower bound and the convexity of the boundary, including the gradient estimations, Harnack inequalities, transportation-cost inequalities and other functional inequalities for diffusion semigroup.

Keywords: **Metric flow, curvature, second fundamental from, transportation-cost inequality, Harnack inequality, coupling**
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1 Introduction

The setting for our work is a differential manifold with boundary carrying a time-dependent metric $\{g_t\}_{t\in[0,T_c)}$. More precisely, on a d-dimensional manifold M with boundary ∂M and a one-parameter $C^{1,\infty}$-family of complete Riemannian metrics $\{g_t\}_{t\in[0,T_c)}$. For simplicity, we take the notations: for $X, Y \in TM$,

$$\mathrm{Ric}_t^Z(X,Y) := \mathrm{Ric}_t(X,Y) - \left\langle \nabla_X^t Z_t, Y \right\rangle_t,$$
$$\mathcal{R}_t^Z(X,Y) := \mathrm{Ric}_t^Z(X,Y) - \frac{1}{2}\partial_t g_t(X,Y),$$

where Ric_t is the Ricci curvature tensor with respect to g_t, $(Z_t)_{t\in[0,T_c)}$ is a C^1-family of vector fields, and $\langle \cdot,\cdot \rangle_t := g_t(\cdot,\cdot)$. Define the second fundamental form of the boundary by

$$\mathbb{I}_t(X,Y) = -\left\langle \nabla_X^t N_t, Y \right\rangle_t, \quad X, Y \in T\partial M,$$

where N_t is the inward unit normal vector field of the boundary associated with g_t. Consider the elliptic operator $L_t := \Delta_t + Z_t$. Let X_t be the reflecting inhomogeneous diffusion process generated by L_t (called reflecting L_t-diffusion process). Assume that X_t is non-explosive before

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T_c. In this paper, we first intend to give the derivative formula of the associated diffusion semigroup motivated by Wang's argument (see [39, Theorem 3.2.1]); then based on it, we present a number of equivalent inequalities involving e.g. the gradient inequalities, transportation-cost inequalities, Harnack inequalities, which are important quantitative measure of the effect of curvature $\mathcal{R}_t^Z$ and $\mathbb{I}_t$ on behavior of the distribution of the reflecting L_t-diffusion process.

When the metric is independent of t, Wang did groundbreaking work on stochastic analysis over Riemannian manifolds with boundary, especially non-convex boundary. For example, characterizing the second fundamental form in [34]; establishing the gradient estimates, transportation-cost inequalities, dimension-free Harnack inequalities, and other functional inequalities on manifold with boundary (see e.g. [34, 36, 37, 38]). Here, we aim to extend the former conclusions to our setting. Compared with constant metric case, $\partial_t g_t$ will become a new important term involved in the curvature condition.

In this paper, we will approach the announced problems from a probabilistic viewpoint. Before moving on, let us briefly recall some known results in the time-inhomogeneous Riemannian setting without boundary. In [1], Coulibaly et al constructed the g_t-Brownian motion (i.e. the diffusion generated by $\frac{1}{2}\Delta_t$), established the Bismut formula when $(g_t)_{t\geq 0}$ is the Ricci flow, which in particular implies the gradient estimates of the associated heat semigroup. Next, by constructing horizontal diffusion processes, Coulibaly [2] investigated the optimal transportation inequality on time-inhomogeneous space. Moreover, Kuwada and Philipowski studied the non-explosion of g_t-Brownian motion in [22] for the super Ricci flow, and the first author developed the coupling method to estimate the gradient of the semigroup in [24]. For more development on the research on stochastic analysis on time-inhomogeneous space. See [23] for reviewing the monotonicity of $\mathcal{L}$-transportation cost from a probabilistic point; see [10, 11] for the stochastic analysis on path space over time-inhomogeneous space; see [12] for corresponding results on time-inhomogeneous space without boundary. We would like to indicate that when the manifold evolves over Ricci flow, which provides an intrinsic family of time-dependent metrics, every calculations seem like in the case of Ricci flat manifold with constant metric.

The rest parts of the paper are organized as follows. In Section 2, we first introduce the reflecting L_t-diffusion processes; then prove the Kolmogrov equations for the associated semigroup. In Sections 3 and 4, we establish the derivative formula of inhomogeneous semigroup, which is further applied to the gradient estimates of the semigroup and characterizing $\mathcal{R}_t^Z$ and $\mathbb{I}_t$. In Section 5, some important inequalities including transportation-cost inequality, Harnack inequalities and other functional inequalities are proved to be equivalent to the lower bound of $\mathcal{R}_t^Z$ and the convexity of the boundary, and in the finial section, we use coupling method to give a probabilistic proof of Harnack inequality and extend it to non-convex case.

2 The reflecting L_t-diffusion process

Let $\mathcal{F}(M)$ be the frame bundle over M and $\mathcal{O}_t(M)$ be the orthonormal frame bundle over M with respect to g_t. Let $\mathbf{p}: \mathcal{F}(M) \to M$ be the projection from $\mathcal{F}(M)$ onto M. Let $\{e_\alpha\}_{\alpha=1}^d$

be the canonical orthonormal basis of $\mathbb{R}^d$. For any $u \in \mathcal{F}(M)$, let $H_i^t(u)$ be the ∇^t horizontal lift of ue_i and $\{V_{\alpha,\beta}(u)\}_{\alpha,\beta=1}^d$ be the canonical basis of vertical fields over $\mathcal{F}(M)$, defined by $V_{\alpha,\beta}(u) = Tl_u(\exp(E_{\alpha,\beta}))$, where $E_{\alpha,\beta}$ is the canonical basis of $\mathcal{M}_d(\mathbb{R})$, the $d \times d$ matric space over $\mathbb{R}$, and $l_u : Gl_d(\mathbb{R}) \to \mathcal{F}(M)$ is the left multiplication from the general linear group to $\mathcal{F}(M)$, i.e. $l_u \exp(E_{\alpha,\beta}) = u \exp(E_{\alpha,\beta})$.

Let $B_t := (B_t^1, B_t^2, \cdots, B_t^d)$ be a $\mathbb{R}^d$-valued Brownian motion on a complete filtered probability space $(\Omega, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ with the natural filtration $\{\mathscr{F}_t\}_{t\geq 0}$. Assume the elliptic generator L_t is a C^1 functional of time with associated metric g_t:

$$L_t = \Delta_t + Z_t$$

where Z_t is a $C^{1,1}$ vector field on M. As in the time-homogeneous case, to construct the L_t-diffusion process, we first construct the corresponding horizontal diffusion process generated by $\Delta_{\mathcal{O}_t(M)} + H_{Z_t}^t$ by solving the Stratonovich stochastic diffusion equation (SDE):

$$\begin{cases} \mathrm{d}u_t = \sqrt{2}\displaystyle\sum_{i=1}^d H_i^t(u_t) \circ \mathrm{d}B_t^i + H_{Z_t}^t(u_t)\mathrm{d}t - \frac{1}{2}\sum_{i,j} \partial_t g_t(u_t e_i, u_t e_j) V_{i,j}(u_t)\mathrm{d}t + H_{N_t}^t(u_t)\mathrm{d}l_t, \\ u_0 \in \mathcal{O}_0(M), \end{cases}$$

where $\Delta_{\mathcal{O}_t(M)}$ is the horizontal Laplace operator on $\mathcal{O}_t(M)$; $H_{Z_t}^t(u_t)$ and $H_{N_t}^t$ are the ∇^t horizontal lift of Z_t and N_t respectively; l_t is an increasing process supported on $\{t \geq 0 : X_t := \mathbf{p}u_t \in \partial M\}$. By a similar discussion as in [1, Proposition 1.2], we see that the last term promises $u_t \in \mathcal{O}_t(M)$. Since $(H_{Z_t}^t)_{t\in[0,T_c)}$ is a $C^{1,1}$-family vector field, it is well-known that (see e.g. [16]) the equation has a unique solution up to the life time $\zeta := \lim_{n\to\infty} \zeta_n$, and

$$\zeta_n := \inf\{t \in [0, T_c) : \rho_t(\mathbf{p}u_0, \mathbf{p}u_t) \geq n\}, \ n \geq 1, \quad \inf \varnothing = T_c,$$

where $\rho_t(x, y)$ is the distance between x and y associated with g_t. Let $X_t = \mathbf{p}u_t$. It is easy to see that X_t solves the equation

$$\mathrm{d}X_t = \sqrt{2}u_t \circ \mathrm{d}B_t + Z_t(X_t)\mathrm{d}t + N_t(X_t)\mathrm{d}l_t, \quad X_0 = x := \mathbf{p}u$$

up to the life time ζ. By the Itô formula, for any $f \in C_0^{1,2}([0, T_c) \times M)$ with $N_t f_t := N_t f_t|_{\partial M} = 0$,

$$f(t, X_t) - f(0, x) - \int_0^t (\partial_s + L_s) f(s, X_s)\mathrm{d}s = \sqrt{2}\int_0^t \left\langle u_s^{-1}\nabla^s f(s, \cdot)(X_s), \mathrm{d}B_s \right\rangle_s$$

is a martingale up to the life time ζ. So, we call X_t the reflecting diffusion process generated by L_t. When $Z_t \equiv 0$, then $\tilde{X}_t := X_{t/2}$ is generated by $\frac{1}{2}\Delta_t$ and is called the reflecting g_t-Brownian motion on M.

2.1 Kolmogorov equations

In this section, we introduce the Kolmogrov equations for $P_{s,t}$, the inhomogeneous semigroup of the reflecting diffusion process generated by L_t. Let

$$\mathscr{C}_N(L) = \{f(t, x) \in C^{1,\infty}([0, T_c) \times M), N_t f_t|_{\partial M} = 0, L_t f_t \in \mathscr{B}_b(M)\}.$$

Here and what follows, in some place, we write $f_t = f(t,\cdot)$ or $g_t = g(t)$ for simplicity. Assume that X_t is the reflecting L_t-diffusion process, we claim that $F(t,x) = P_{t,T}f(x)$ is the unique solution to the Neumann heat equation

$$\partial_t F(t,x) = -L_t F(t,x), \quad N_t F(t,\cdot) = 0 \ \text{ for all } t \in [0,T], \ F(T,\cdot) = f. \tag{2.1}$$ cauchy2

To this end, we need the following two lemmas and they are essentially due to [34].

211 **Lemma 2.1.** *For any $x \in M$ and $r_0 > 0$, and X_t is the reflecting L_t-diffusion process with $X_0 = x$, Then,*

(1) *for any $t_0 \in [0, T_c)$, there exists a constant $c > 0$ such that*

$$\mathbb{P}^x(\sigma_r \le t) \le e^{-cr^2/t}, \ r \in [0, r_0], \ t \in [0, 1 \wedge T_c]$$

holds, where $\sigma_r = \inf\{s \ge 0 : \rho_{t_0}(X_s, x) \ge r\}$.

(2) *there exists a constant $c_1 > 0$ such that*

$$\mathbb{P}^x(\tilde\sigma_r \le t) \le e^{-c_1 r^2/t}, \ r \in [0, r_0], \ t \in [0, 1 \wedge T_c]$$

holds, where $\tilde\sigma_r = \inf\{s \ge 0 : \rho_s(X_s, x) \ge r\}$.

Proof. For (1), since we use the Itô formula for $\rho_{t_0}(x, X_t)$, where the metric is fixed, the proof is similar as in [34, Proposition A.2], we omit it here. Note that when $(\partial M, g_t)$ is non-convex for some $t \in [0, T_c)$, let $\phi \in C_b^{1,\infty}([0,T_c) \times M)$ such that for any $t \in [0, T_c)$, $\phi_t := \phi(t,\cdot) \ge 1$ and ∂M is convex in $B_t(x, r_0)$ under $\tilde g_t := \phi_t^{-2} g_t$, where $B_t(x, r_0)$ is the g_t-geodesic ball with center x and radius r_0. For the existence of ϕ, see e.g. [33].

For (2), it also leaves us to prove the convex boundary case. Note that there exists a constant $C_1 > 0$ such that

$$L_t \rho_t^2(x,\cdot)(y) = 2\rho_t(x,y) L_t \rho_t(x,\cdot)(y) + 2\rho_t(x,y)\partial_t \rho_t(x,y) + 2 \le C_1$$

holds on $\bigcup_{t \in [0,1]} \{t\} \times B_t(x,r)$. Due to this inequality, the following discussion is similar as the case with constant metric. □

212 **Lemma 2.2.** *Let $x \in \partial M$, and let $\tilde\sigma_r$ be the same as in Lemma 2.1 for a fixed $r > 0$. Then*

(1) $\mathbb{E}^x e^{\lambda l_{t\wedge\tilde\sigma_r}} < \infty$, *for any $\lambda > 0$.*

(2) $\mathbb{E}^x l_{t\wedge\tilde\sigma_r} = \frac{2\sqrt{t}}{\sqrt{\pi}} + O(t^{3/2})$ *holds for small $t > 0$.*

Proof. Let $h \in C_0^{1,\infty}([0,T_c) \times M)$ be the non-negative such that $h(t,\cdot)|_{\partial M} = 0$ and $N_t h_t = 1$ holds on $D := \{(t,x) : t \in [0,T_c), x \in B_t(x,r)\}$. Since ρ_t^∂ , the g_t-distance to the boundary, is smooth on a neighborhood of ∂M. h_t can be constructed such that $h_t = \rho_t^\partial$ in a neighborhood of $\partial M \cap B_t(x,r)$. Note that $(L_t + \partial_t)h(t,x)$ is bounded on D, the remainder of the proof is similar to that of [34, Theorem 2.1]. □

2t1 **Theorem 2.3.** *For $f \in \mathscr{C}_N(L)$, the following forward Kolmogorov equation holds,*

$$\frac{\mathrm{d}}{\mathrm{d}t} P_{s,t} f(t,x) = P_{s,t}\left(L_t f + \partial_t f\right)(t,x), \quad 0 \le s < t < T_c. \tag{2.2}$$

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For $f \in \mathscr{B}_b(M)$, there hold

(1) *the backward Kolmogrov equation*

$$\frac{\mathrm{d}}{\mathrm{d}s} P_{s,t} f = -L_s P_{s,t} f;$$

(2) $N_s P_{s,t} f = 0$, *for all* $0 \le s < t < T_c$*;*

(3) *let* $0 < t < T_c$ *and* $\psi \in C_b^2([\inf f, \sup f])$*. If* $|\nabla^{\cdot} P_{\cdot,t} f|_{\cdot}$ *is bounded on* $[0,t] \times M$*. Then*

$$\frac{\mathrm{d}}{\mathrm{d}s} P_{0,s} \psi(P_{s,t} f) = P_{0,s}\left(\psi''(P_{s,t} f) |\nabla^s P_{s,t} f|_s^2\right), \quad s \in [0,t].$$

Proof. (a) The first assertion follows from

$$P_{s,t} f(t,x) = f(s,x) + \int_s^t P_{s,r}\left(L_r f + \partial_r f\right)(r,x) \mathrm{d}r.$$

To prove (1), it suffices to consider $x \in M^\circ := M \setminus \partial M$ and $s = 0$. Let $r_0 > 0$ be such that $B_0(x, r_0) \in M^\circ$, and take $h \in C_0^\infty(M)$ such that

$$h|_{B_0(x, r_0/2)} = 1, \ \text{and } h|_{B_0(x,r_0)^c} = 0.$$

By the Itô formula, we have

$$\mathrm{d}(h P_{s,t} f)(X_s) = \mathrm{d}M_s + \left\{ L_s(h P_{s,t} f) + h \frac{\mathrm{d}}{\mathrm{d}s} P_{s,t} f \right\}(X_s) \mathrm{d}s.$$

Then,

$$\begin{aligned} \lim_{s \downarrow 0} \frac{\mathbb{E}^x(h P_{s,t} f(X_s)) - P_{0,t} f(x)}{s} &= \lim_{s \downarrow 0} \mathbb{E}^x \frac{1}{s} \int_0^s \left\{ L_r(h P_{r,t} f) + h \frac{\mathrm{d}}{\mathrm{d}r} P_{r,t} f \right\}(X_r) \mathrm{d}r \\ &= \left\{ L_r(P_{r,t} f) + \frac{\mathrm{d}}{\mathrm{d}r} P_{r,t} f|_{r=0} \right\}(x). \end{aligned} \tag{2.3}$$

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On the other hand, by Lemma 2.1,

$$\mathbb{E}^x(h P_{s,t} f)(X_s) - P_{0,t} f(x) = \mathbb{E}^x((h-1) P_{s,t} f)(X_s) \le \|f(h-1)\|_\infty e^{-c/s}, \ s \in (0,1]$$

holds for some constant $c > 0$. Combining this with (2.3), we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}r} P_{r,t} f(x)\big|_{r=0} = -L_r P_{r,t} f(x).$$

(b) Let $x \in M$. If $N_s P_{s,t} f \neq 0$, $0 \le s < t < T_c$. For instance $N_0 P_{0,t} f(x) > 0$, there exist $r_0 > 0$, small $t > t_0 > 0$ and $\varepsilon > 0$, such that $N_r P_{r,t} f(x) > \varepsilon$ holds on $B_r(s, 2r_0)$ and $r \in (0, t_0)$. Moreover, we assume $f \ge 0$. Let $h \in C^\infty([0,T_c) \times M)$, $0 \le h \le 1$ such that

$$N_r h_r = 0, \quad h_r|_{B_r(x,r_0)} = 1 \ \text{ and } \ h_r|_{B_r(x,2r_0)^c} = 0.$$

By the Itô formula,

$$
\begin{aligned}
\mathbb{E}^x\left(h_s P_{s,t} f(X_s)\right) =& P_{0,t} f(x) + \int_0^s P_{0,r}\left[L_r(h_r P_{r,t} f) + \frac{\mathrm{d}h_r}{\mathrm{d}r}\cdot P_{r,t} f + h_r \frac{\mathrm{d}}{\mathrm{d}r} P_{r,t} f\right](x)\mathrm{d}r \\
&+ \mathbb{E}^x \int_0^s h_r N_r P_{r,t} f(X_r)\mathrm{d}l_r. \qquad (2.4) \quad \boxed{\texttt{2eq3}}
\end{aligned}
$$

Moreover, by Lemma 2.1(2),

$$
\begin{aligned}
\mathbb{E}^x(h_s P_{s,t} f)(X_s) - P_{0,t} f(x) =& \mathbb{E}^x((h-1)P_{s,t} f)(X_s) \le \|f(h-1)\|_\infty \mathbb{P}^x(\tilde{\sigma}_{r_0} \le s) \\
\le& \|f(h-1)\|_\infty e^{-c/s}, \ \ s \in (0,1]
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{s\downarrow 0} \frac{1}{s}\int_0^s P_{0,r}\left[L_r(h_r P_{r,t} f) + \frac{\mathrm{d}h_r}{\mathrm{d}r}\cdot P_{r,t} f + h_r \frac{\mathrm{d}}{\mathrm{d}r} P_{r,t} f\right](x)\mathrm{d}r \\
=& L_0 P_{0,t} f + \frac{\mathrm{d}}{\mathrm{d}r}\Big|_{r=0} P_{r,t} f + \partial_r\big|_{r=0} h(r,x) = 0.
\end{aligned}
$$

The last equality comes from (1) and $h(r,x) = 1$. Combining this with (2.4) we arrive at

$$
\varepsilon \lim_{s\to 0} \frac{1}{s}\mathbb{E}^x l_{s\wedge\tilde{\sigma}_{r_0}} \le 0,
$$

which is impossible according to Lemma 2.2.

(c) Combining (1), (2) and the Itô formula, there is a local martingale M_s such that

$$
\begin{aligned}
\mathrm{d}\psi(P_{s,t} f)(X_s) =& \mathrm{d}M_s + \{L_s \psi(P_{s,t} f) - \psi'(P_{s,t} f) L_s P_{s,t} f\}(X_s)\mathrm{d}s \\
=& \mathrm{d}M_s + \{\psi''(P_{s,t} f)|\nabla^s P_{s,t} f|_s^2\}(X_s)\mathrm{d}s, \quad s \in [0,t],
\end{aligned}
$$

where

$$
\mathrm{d}M_s = \sqrt{2}\left\langle \nabla^s \psi(P_{s,t} f), u_s \mathrm{d}B_s\right\rangle_s, \quad s \in [0,t].
$$

Since $|\nabla^\cdot P_{\cdot,t}|_\cdot$ is bounded on $[0,t]\times M$ and $\psi \in C_b^2([\inf f, \sup f])$, M_s is a martingale. Therefore,

$$
P_{0,s}\psi(P_{s,t} f) = \mathbb{E}\psi(P_{s,t} f)(X_s) = \psi(P_{0,t} f) + \int_0^s P_{0,r}\{\psi''(P_{s,t} f)|\nabla^t P_{s,t} f|_s^2\}\mathrm{d}r.
$$

We end the proof. $\square$

By Theorem 2.3, we see that $P_{t,T} f$ is the unique solution of (2.1). Consider the usual Neumann problem,

$$
\begin{cases}
\partial_t u(t,x) = L_t u(t,x), & x \in M, \ t \in [0,T], \\
u(0,x) = f(x), & x \in M, \\
N_t u(t,\cdot) = 0, & x \in \partial M, \ t \in (0,T].
\end{cases} \qquad (2.5) \quad \boxed{\texttt{cauchy}}
$$

The following result give the unique solution of it.

$\boxed{\texttt{2c1}}$ **Corollary 2.4.** *Let X_t^T be a reflecting $L_{(T-t)}$-diffusion process. Suppose that $X_\cdot^T$ is non-explosive before T. Let $\overline{P}_{s,t}$ be the inhomogeneous semigroup. For some $f \in \mathscr{B}_b(M)$, $F(t,x) := \overline{P}_{T-t,T} f(x)$ is the unique C^2-solution to (2.5).*

3 Formulae for $\mathcal{R}_t^Z$ and $\mathbb{I}_t$

In this section, we first give the derivative formula for the Neumann semigroup, which extends the result of Wang [39, Theorem 3.2.1] or Hsu [17, Theorem 5.1] when the metric is independent of t and $\partial M \neq \varnothing$, and Cheng [12, Theorem 3.1] for time-inhomogeneous manifold without boundary; then apply it to characterizing $\mathcal{R}_t^Z$ and $\mathbb{I}_t$.

3.1 Derivative formula

First, we introduce some basic notations. For $u \in \mathcal{F}(M)$, let $\mathcal{R}_t^Z(u)$ and $\mathbb{I}_t(u)$ be $\mathbb{R}^d \otimes \mathbb{R}^d$-valued random variables such that

$$\mathcal{R}_t^Z(u)(a,b) = \mathcal{R}_t^Z(ua, ub),\ \ \mathbb{I}_t(u)(a,b) = \mathbb{I}_t(\mathbf{P}_\partial ua, \mathbf{P}_\partial ub),\ \ a, b \in \mathbb{R}^d,$$

where for any $z \in \partial M$, $\mathbf{P}_\partial : T_zM \to T_z\partial M$ is the projection operator. Our main result in this section is presented as follows.

Bis **Theorem 3.1.** *Let $0 \le s < t < T_c$ and $u_0 \in \mathcal{O}_0(M)$ be fixed, and let $K \in C([0,T_c) \times M)$ and $\sigma \in C([0,T_c) \times \partial M)$ be such that $\mathcal{R}_t^Z \ge K(t,\cdot)$ and $\mathbb{I}_t \ge \sigma(t,\cdot)$. Assume that*

$$\sup_{u\in[s,t]} \mathbb{E}^x \exp\left[-\int_s^u K(r,X_r)\mathrm{d}r - \int_s^u \sigma(r,X_r)\mathrm{d}l_r\right] < \infty. \tag{3.1}$$ 2qt

Then there exists a progressively measurable process $\{Q_{s,r}\}_{0\le s\le r<T_c}$ on $\mathbb{R}^d \otimes \mathbb{R}^d$ such that

$$Q_{s,s} = I, \quad \|Q_{s,r}\| \le \exp\left[-\int_s^r K(u,X_u)\mathrm{d}u - \int_s^r \sigma(u,X_u)\mathrm{d}l_u\right], \quad r \in [s,t],$$

and for any $f \in C_b^1(M)$ such that $|\nabla^s P_{s,\cdot} f|_s$ is bounded on $[s,t]\times M$, and $h \in C^1((s,t))$ satisfying $h(s) = 0, h(t) = 1$, there holds

$$\begin{aligned}(u_s)^{-1}\nabla^s P_{s,t}f(x) &= \mathbb{E}\left\{Q_{s,t}^* u_t^{-1}\nabla^t f(X_t)\big|X_s = x\right\}\\ &= \frac{1}{\sqrt{2}}\mathbb{E}\left\{f(X_t)\int_s^t h'(r)Q_{s,r}^*\mathrm{d}B_r\big|X_s = x\right\}.\end{aligned} \tag{3.2}$$ 2Bis

Proof. Without loss generality, we assume $s = 0$. And simply denote $Q_{0,t}$ by Q_t. The essential idea is due to [16, Theorem 4.2] for the case with constant metric.

(1) Construction of Q_s. For any $n \ge 1$, let $Q_s^{(n)}$ solve the equation

$$\begin{cases}\mathrm{d}Q_s^{(n)} = -\mathcal{R}_s^Z(u_s)Q_s^{(n)}\mathrm{d}s - \mathbb{I}_s(u_s)Q_s^{(n)}\mathrm{d}l_s\\ \qquad\qquad -\frac{1}{2}(n + 2(\sigma(s,X_s))^+)\left((Q_s^{(n)})^* u_s^{-1}N_s\right) \otimes \left(u_s^{-1}N_s\right)\mathrm{d}l_s,\\ Q_0 = I.\end{cases}$$

It is easy to see that for any $a \in \mathbb{R}^d$,

$$\begin{aligned}\mathrm{d}\|Q_s^{(n)}a\|^2 &= 2\left\langle \mathrm{d}Q_s^{(n)}a, Q_s^{(n)}a\right\rangle_{\mathbb{R}^d}\\ &= -2\mathcal{R}_s^Z(u_sQ_s^{(n)}a, u_sQ_s^{(n)}a)\mathrm{d}s - 2\mathbb{I}_s(\mathbf{P}_\partial u_sQ_s^{(n)}a, \mathbf{P}_\partial u_sQ_s^{(n)}b)\mathrm{d}l_s\\ &\quad - (n + 2\sigma(s,X_s)^+)\left\langle u_sQ_s^{(n)}a, N_s\right\rangle_s^2 \mathrm{d}l_s\\ &\le -2\|Q_s^{(n)}a\|^2\left[K(s,X_s)\mathrm{d}s + \sigma(s,X_s)\mathrm{d}l_s\right] - n\left\langle u_sQ_s^{(n)}a, N_s\right\rangle_s^2 \mathrm{d}l_s,\end{aligned}$$

where $\|\cdot\|$ is the operator norm on $\mathbb{R}^d$. Therefore,

$$\|Q_s^{(n)}\|^2 \le \exp\left[-2\int_0^s K(r,X_r)\mathrm{d}r - 2\int_0^s \sigma(r,X_r)\mathrm{d}l_r\right] < \infty, \tag{3.3}$$

and for any $m \ge 1$,

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}^x \int_0^{t\wedge\zeta_m} \|(Q_s^{(n)})^* u_s^{-1} N_s\|^2 \mathrm{d}l_s \\
&\le \lim_{n\to\infty}\left(\frac{1}{n} + \frac{1}{n}\mathbb{E}^x \int_0^{t\wedge\xi_m} 2\|Q_s^{(n)}\|^2 \left[|K|(s,X_s)\mathrm{d}s + |\sigma|(s,X_s)\mathrm{d}l_s\right]\right) = 0, \qquad (3.4)
\end{aligned}$$

where the second equality follows from Lemma 2.1 (2), (3.3) and the boundedness of K and σ on $B_t(x,m)$. Combining this with (3.3) and (3.1), we see that

$$\sup_{n\ge1}\left\{\mathbb{E}^x\int_0^t \|Q_s^{(n)}\|\mathrm{d}s + \mathbb{E}^x\|Q_t^{(n)}\|\right\} < \infty.$$

There exists a subsequence $\{Q^{n_k}\}$ and a progressively measurable process Q such that for any bounded measurable process $(\varphi_s)_{s\in[0,t]}$ on $\mathbb{R}^d$ and any $\mathbb{R}^d$-valued random variable η, there holds

$$\lim_{n\to\infty}\left\{\mathbb{E}^x\int_0^t (Q_s^{(n_k)} - Q_s)\varphi_s \mathrm{d}s + \mathbb{E}^x(Q_t^{n_k} - Q_t)\eta\right\} = 0.$$

(2) Proof of the first equality. Following form the proof of [12, Theorem 3.1], we have

$$\begin{aligned}
\mathrm{d}(\mathbf{d}P_{s,t}f)(X_s) =& \nabla^s_{u_s\mathbf{d}B_s}(\mathbf{d}P_{s,t}f)(X_s) + \mathrm{Ric}_s^Z(\cdot,\ \nabla^s P_{s,t}f(X_s))\mathrm{d}s \\
&+ \nabla^s_{N_s}(\mathbf{d}P_{s,t}f)(X_s)\mathrm{d}l_s. \qquad (3.5)
\end{aligned}$$

Now for any $a \in \mathbb{R}^d$,

$$\begin{aligned}
D^s u_s Q_s^{(n)} a =& -\mathrm{Ric}_s^Z(u_s Q_s^{(n)} a, \cdot)\mathrm{d}s - \mathbb{I}_s(\mathbf{P}_\partial u_s Q_s^{(n)} a, \cdot)\mathrm{d}l_s \\
&- \frac{1}{2}(n + 2\sigma(s,X_s))^+ \left\langle N_s, u_s Q_s^{(n)} a\right\rangle_s \langle N_s, \cdot\rangle_s \mathrm{d}l_s.
\end{aligned}$$

Then by Lemma 2.2(2),

$$\begin{aligned}
\mathrm{d}(\mathbf{d}P_{s,t}f)(X_s)(u_s Q_s^{(n)} a) =& \mathrm{Hess}^s_{P_{s,t}f}(u_s Q_s^{(n)} a, u_s \mathrm{d}B_s) + \mathrm{Hess}^s_{P_{s,t}f}(u_s Q_s^{(n)} a, N_s)\mathrm{d}l_s \\
&- \mathbb{I}_s(\mathbf{P}_\partial u_s Q_s^{(n)} a, \nabla^s P_{s,t}f(X_s))\mathrm{d}l_s.
\end{aligned}$$

Since for any $v \in T_z\partial M$, $z \in \partial M$, we have

$$0 = v\left\langle N_s, \nabla^s P_{s,t}f\right\rangle_s(z) = \left\langle \nabla^s_v N_s, \nabla^s P_{s,t}f\right\rangle_s(z) + \mathrm{Hess}^s_{P_{s,t}f}(v, N_s),$$

which implies

$$\mathrm{Hess}^s_{P_{s,t}f}(v, N_s) = \mathbb{I}_s(v, \nabla^s P_{s,t}f)(z).$$

We arrive at

$$\begin{aligned}
&\mathrm{d}\left\langle \nabla^s P_{s,t}f(X_s), u_s Q_s^{(n)} a\right\rangle_s \\
&= \mathrm{Hess}^s_{P_{s,t}f}(u_s Q_s^{(n)} a, u_s \mathrm{d}B_s) + \mathrm{Hess}^s_{P_{s,t}f}(N_s, N_s)\left\langle u_s Q_s^{(n)} a, N_s\right\rangle_s \mathrm{d}l_s.
\end{aligned}$$

Combining this with (3.4) and the boundedness of $|\nabla^{\cdot} P_{\cdot,t} f|_{\cdot}$ on $[0,t]\times M$, we obtain

$$\begin{aligned}
\left\langle \nabla^0 P_{0,t} f, u_0 a \right\rangle_0 &= \lim_{m\to\infty} \lim_{k\to 0} \mathbb{E}^x \left\langle \nabla^{t\wedge\zeta_m} P_{t\wedge\zeta_m,t} f(X_{t\wedge\zeta_m}), u_{t\wedge\zeta_m} Q^{(n_k)}_{t\wedge\zeta_m} a \right\rangle_{t\wedge\zeta_m} \\
&= \lim_{m\to\infty} \lim_{k\to\infty} \mathbb{E}^x \left\{ \mathbf{1}_{\{t\le\zeta_m\}} \left\langle \nabla^t f(X_t), u_t Q^{(n_k)}_t a \right\rangle_t \right\} \\
&= \mathbb{E}^x \left\langle \nabla^t f(X_t), u_t Q_t a \right\rangle_t .
\end{aligned}$$

This implies the first equality.

(3) Proof of the second equality. Since by the Itô formula, we obtain

$$\mathrm{d}P_{s,t} f(X_s) = \sqrt{2}\, \langle \nabla^s f(X_s), u_s \mathrm{d}B_s \rangle_s .$$

Therefore, we have

$$f(X_t) = P_{0,t} f(x) + \sqrt{2} \int_0^t \langle \nabla^s P_{s,t} f(X_s), u_s \mathrm{d}B_s \rangle_s .$$

So for any $a\in\mathbb{R}^d$ and $m\ge 1$, it follows from (3.3), (3.4) and the boundedness of $\{|\nabla^s P_{s,t} f|_r\}_{r\in[s,t]}$ that

$$\begin{aligned}
&\frac{1}{\sqrt{2}} \mathbb{E}^x \left\{ f(X_t) \int_0^t h'(s) \langle Q_s a, \mathrm{d}B_s \rangle_{\mathbb{R}^d} \right\} = \mathbb{E}^x \left\{ \int_0^t h'(s) \langle u_s Q_s a, \nabla^s P_{s,t} f(X_s) \rangle_s \, \mathrm{d}s \right\} \\
&= \lim_{k\to\infty} \mathbb{E}^x \left\{ \int_0^t h'(s) \left\langle u_s Q^{(n_k)}_s a, \nabla^s P_{s,t} f(X_s) \right\rangle_s \mathrm{d}s \right\} \\
&= \lim_{m\to\infty} \lim_{k\to\infty} \int_0^t h'(s) \mathbb{E}^x \left\langle u_{s\wedge\zeta_m} Q^{(n_k)}_{s\wedge\zeta_m}, \nabla^{s\wedge\zeta_m} P_{s\wedge\zeta_m,t} f(X_{s\wedge\zeta_m}) \right\rangle_{s\wedge\zeta_m} \mathrm{d}s \\
&= \int_0^t h'(s) \langle u_0 a, \nabla^0 P_{0,t} f(x) \rangle_0 \, \mathrm{d}s \\
&= \left\langle \nabla^0 P_{0,t} f(x), u_0 a \right\rangle_0 .
\end{aligned}$$

We complete the proof. $\square$

Next, combining the above argument with the proof of Theorem 3.1, we have the following local version of derivative formula of $P_{s,t}$.

2Bismut **Theorem 3.2.** *Let $0\le s\le t<T_c$. Assume $\mathcal{R}^Z_r \ge K(r,\cdot)$ and $\mathbb{I}_r \ge \sigma(r,\cdot)$, for some $K\in C([0,T_c)\times M)$ and $\sigma\in C([0,T_c)\times\partial M)$. Let $x\in M$ and D be a compact domain of $[s,t]\times M$ such that $(s,x)\in D^\circ$, let τ_D be the first hitting times of (t,X_t) to ∂D, where $X_s=x$. Then there exists a progressively measurable process $\{Q_{s,r}\}_{r\in[s,t]}$ on $\mathbb{R}^d\otimes\mathbb{R}^d$ with*

$$\|Q_{s,r}\| \le \exp\left[-\int_s^{r\wedge\tau_D} K(u,X_u)\mathrm{d}u - \int_s^{r\wedge\tau_D} \sigma(u,X_u)\mathrm{d}l_u \right], \text{ for all } 0\le s\le r\le t$$

such that for any $\mathbb{R}_+$-valued process h satisfying $h(s)=0$, $h(r)=1$ for $r>t\wedge\tau_D$ and $\mathbb{E}(\int_s^t h'(r)^2\mathrm{d}r)^\alpha<\infty$ for some $\alpha>1/2$, there holds:

$$u_s^{-1}\nabla^s P_{s,t} f(x) = \frac{1}{\sqrt{2}} \mathbb{E}\left\{ P_{t\wedge\tau_D,t} f(X_{t\wedge\tau_D}) \int_s^t h'(r) Q^*_{s,r} \mathrm{d}B_r \big| X_s = x \right\}, \quad f\in\mathscr{B}_b(M).$$

3.2 Formulae for $\mathcal{R}_t^Z$ and $\mathbb{I}_t$

Due to the Derivative formula, we have the following characters about $\mathcal{R}_t^Z$ and $\mathbb{I}_t$ with a similar discussion as that in the case with constant metric, where the formulae for $\mathcal{R}_t^Z$ is essentially due to [6] and [5, Propostions 2.1 and 2.6]; the formulae for second fundamental form were first proved by Wang[35].

4t1 **Theorem 3.3.** *For $s \in [0, T_c)$, let $x \in M^\circ$ and $X \in T_xM$ with $|X|_s = 1$. Let $f \in C_0^\infty(M)$ such that $N_s f|_{\partial M} = 0$ and $\nabla^s f = X$, and let $f_n = f + n$. Then,*

(1) *for any $p > 0$,*

$$\mathcal{R}_s^Z(X, X) = \lim_{t \downarrow s} \frac{P_{s,t}|\nabla^t f|_t^p(x) - |\nabla^s P_{s,t} f|_s^p(x)}{p(t-s)}; \tag{3.6}$$ 1-2

(2) *for any $p > 1$,*

$$\begin{aligned}\mathcal{R}_s^Z(X, X) &= \lim_{n\to\infty} \lim_{t\downarrow s} \frac{1}{t-s} \left(\frac{p\{P_{s,t} f_n^2 - (P_{s,t} f_n^{\frac{2}{p}})^p\}}{4(p-1)(t-s)} - |\nabla^s P_{s,t} f_n|_s^2 \right)(x) \\ &= \lim_{n\to\infty} \lim_{s\downarrow t} \frac{1}{t-s} \left(P_{s,t}|\nabla^t f|_t^2 - \frac{p\{P_{s,t} f_n^2 - (P_{s,t} f_n^{\frac{2}{p}})^p\}}{4(p-1)(t-s)} \right)(x); \end{aligned} \tag{3.7}$$ 2-2

(3) *$\mathcal{R}_s^Z(X, X)$ is equal to each of the following limits:*

$$\lim_{n\to\infty} \lim_{t\downarrow s} \frac{1}{(t-s)^2} \left\{ (P_{s,t} f_n) \left[P_{s,t}(f_n \log f_n) - (P_{s,t} f_n) \log P_{s,t} f_n \right] - (t-s) |\nabla^s P_{s,t} f|_s^2 \right\}(x). \tag{3.8}$$ 4-2

$$\lim_{n\to\infty} \lim_{t\downarrow s} \frac{1}{4(t-s)^2} \left\{ 4(t-s) P_{s,t}|\nabla^t f|_t^2 + (P_{s,t} f_n^2) \log P_{s,t} f_n^2 - P_{s,t} f_n^2 \log f_n^2 \right\}(x). \tag{3.9}$$ 5-2

Proof. Let $r > 0$ be such that $B_0(x, r) \subset M^\circ$ and $|\nabla^t f|_t \geq \frac{1}{2}$. Due to Lemma 2.1(1), the proof of [12, Theorem 3.4] also works for the present setting by replacing s by $s \wedge \sigma_r$, where $\sigma_r := \inf\{s > t : X_s \notin B_0(x, r)\}$. So that the boundary condition need not to be considered. □

II-form **Theorem 3.4.** *For $s \in [0, T_c)$, let $x \in \partial M$ and $X \in T_xM$ with $|X|_s = 1$. Then for any $f \in C_0^\infty(M)$ such that $\nabla^s f(x) = X$,*

$$\begin{aligned}\mathbb{I}_s(X, X) &= \lim_{t\downarrow s} \frac{\pi}{2p\sqrt{t-s}} \left\{ P_{s,t}|\nabla^t f|_t^p - |\nabla^s f|_s^p \right\}(x) \\ &= \lim_{t\downarrow s} \frac{\pi}{2p\sqrt{t-s}} \left\{ P_{s,t}|\nabla^t f|_t^p - |\nabla^s P_{s,t} f|_s^p \right\}(x), \ \ p > 0.\end{aligned}$$

If moreover $f > 0$, then for $p \in [1, 2]$,

$$\begin{aligned}\mathbb{I}_s(X, X) &= -\lim_{t\downarrow s} \frac{3\sqrt{\pi}}{8\sqrt{t-s}} \left\{ |\nabla^s f|_s^2 + \frac{p\{(P_{s,t} f^{2/p})^p - P_{s,t} f^2\}}{4(p-1)(t-s)} \right\}(x) \\ &= -\lim_{t\downarrow s} \frac{3\sqrt{\pi}}{8\sqrt{t-s}} \left(|\nabla^s P_{s,t} f|_s^2 + \frac{p\{(P_{s,t} f^{2/p})^p - P_{s,t} f^2\}}{4(p-1)(t-s)} \right)(x),\end{aligned}$$

where when $p = 1$, we set

$$\begin{aligned}\frac{(P_{s,t} f^{2/p} - P_{s,t} f^2)^p}{p-1} &= \lim_{p\downarrow 1} \frac{(P_{s,t} f^{2/p})^p - P_{s,t} f^2}{p-1} \\ &= (P_{s,t} f^2) \log P_{s,t} f^2 - P_{s,t}(f^2 \log f^2).\end{aligned}$$

Proof. By using Lemma 2.1 and Lemma 2.2, the proof is similar as in time-homogeneous case. We refer the reader to [35, Theorem 1.2] or [39, Theorem 3.2.4] for details. $\square$

4 Coupling for reflecting L_t-diffusion process and gradient estimation

In this section, we first introduce the coupling for the reflecting L_t-diffusion processes and then apply to the gradient estimations of the inhomogeneous semigroup. Let $\mathrm{Cut}_t(x)$ be the set of the g_t cut-locus of x on M. Then, the g_t cut-locus Cut_t and the space time cut-locus $\mathrm{Cut}_{\mathrm{ST}}$ are defined by

$$\mathrm{Cut}_t = \{(x,y) \in M \times M \mid y \in \mathrm{Cut}_t(x)\};$$
$$\mathrm{Cut}_{\mathrm{ST}} = \{(t,x,y) \in [0,T_c) \times M \times M \mid (x,y) \in \mathrm{Cut}_t\}.$$

Set $D(M) := \{(x,x)|x \in M\}$. For a smooth curve γ and smooth vector fields $U,\ V$ along γ, the index form $I_t^\gamma(U,V)$ is given by

$$I_t^\gamma(U,V) = \int_\gamma \left(\left\langle \nabla_{\dot\gamma}^t U, \nabla_{\dot\gamma}^t V \right\rangle_t - \left\langle R^t(U,\dot\gamma)\dot\gamma, V \right\rangle_t \right)(\gamma(s))\mathrm{d}s,$$

where R^t is the Ricci tensor with respect to g_t.

For $(x,y) \notin \mathrm{Cut}_t$, let $\{J_i^t\}_{i=1}^{d-1}$ be Jacobi fields along the minimal geodesic γ from x to y with respect to g_t such that at x and y, $\{J_i^t, \dot\gamma : 1 \le i \le d-1\}$ is an orthonormal basis. Let

$$I_t^Z(x,y) := \sum_{i=1}^{d-1} I_t^\gamma(J_i^t, J_i^t) + Z_t\rho_t(\cdot,y)(x) + Z_t\rho_t(x,\cdot)(y).$$

Moreover, let $P_{x,y}^t : T_xM \to T_yM$ be the g_t-parallel transform along the geodesic γ, and let

$$M_{x,y}^t : T_xM \to T_yM;\ v \mapsto P_{x,y}^t v - 2\langle v,\dot\gamma\rangle_t(x)\dot\gamma(y)$$

be the mirror reflection w.r.t. g_t. Then $P_{x,y}^t$ and $M_{x,y}^t$ are smooth outside Cut_t and $D(M)$. For convenience, we set $P_{x,x}^t$ and $M_{x,x}^t$ be the identity for $x \in M$.

To apply the derivative formula in Theorem 3.1, we have to verify in advance the boundedness of $|\nabla^{\cdot} P_{\cdot,t} f|_{\cdot}$ on $[0,t] \times M$. So, we first present a sufficient condition for it. To this end, we shall make use of the coupling by parallel displacement. If $(\partial M, g_t)$ keep convex on $[0,T_c)$, named convex flow. we have $N_t\rho_t(o,\cdot)|_{\partial M} \le 0$ for any inner point $o \in M$, and the distance between two different points can be reached by the minimal geodesic w.r.t. g_t. Therefore, by a similar discussion as in [12, Theorem 4.2], we have the following result.

Theorem 4.1. *Assume $\mathbb{I}_t \ge 0,\ t \in [0,T_c)$. Let $x \ne y$ and $0 < T < T_c$ be fixed. Let $U : [0,T] \times M \times M \to TM^2$ be C^1-smooth in $(\mathrm{Cut}_{\mathrm{ST}} \cup [0,T] \times D(M))^c$.*

(1) *There exist two Brownian motion B_t and $\tilde B_t$ on the probability space $(\Omega, \{\mathscr{F}_t\}_{t\ge0}, \mathbb{P})$ such that*

$$\mathbf{1}_{\{(X_t,\tilde X_t)\notin \mathrm{Cut}_t\}}\mathrm{d}\tilde B_t = \mathbf{1}_{\{(X_t,\tilde X_t)\notin \mathrm{Cut}_t\}} \tilde u_t^{-1} P_{X_t,\tilde X_t}^t u_t \mathrm{d}B_t$$

holds, where X_t with lift u_t and $\tilde{X}_t$ with lift $\tilde{u}_t$ solve the equation

$$\begin{cases} \mathrm{d}X_t = \sqrt{2}u_t \circ \mathrm{d}B_t + Z_t(X_t)\mathrm{d}t + N_t(X_t)\mathrm{d}l_t, & X_0 = x, \\ \mathrm{d}\tilde{X}_t = \sqrt{2}\tilde{u}_t \circ \mathrm{d}\tilde{B}_t + \left\{ Z_t(\tilde{X}_t) + U(t, X_t, \tilde{X}_t)\mathbf{1}_{\{X_t \neq \tilde{X}_t\}} \right\} \mathrm{d}t + N_t(\tilde{X}_t)\mathrm{d}\tilde{l}_t, & \tilde{X}_0 = y, \end{cases} \tag{4.1}$$ 3e4

where $\tilde{l}_t$ is the local time for $\tilde{X}_t$. Moreover,

$$\begin{aligned} \mathrm{d}\rho_t(X_t, \tilde{X}_t) \le \Big\{ &\frac{1}{2}\int_\gamma \partial_t g_t(\dot{\gamma}(s), \dot{\gamma}(s))\mathrm{d}s + I_t^Z(X_t, \tilde{X}_t) \\ &+ \left\langle U(t, X_t, \tilde{X}_t), \nabla^t \rho_t(X_t, \cdot)(\tilde{X}_t) \right\rangle_t \mathbf{1}_{\{X_t \neq \tilde{X}_t\}} \Big\} \mathrm{d}t. \end{aligned} \tag{4.2}$$ 3e5

(2) *The first assertion in (1) holds with $M^t_{X_t, \tilde{X}_t}$ in place of $P^t_{X_t, \tilde{X}_t}$. In this case,*

$$\begin{aligned} \mathrm{d}\rho_t(X_t, \tilde{X}_t) \le 2\sqrt{2}\mathrm{d}b_t + \Big\{ &\frac{1}{2}\int_\gamma \partial_t g_t(\dot{\gamma}(s), \dot{\gamma}(s))\mathrm{d}s + I_t^Z(X_t, \tilde{X}_t) \\ &+ \left\langle U(t, X_t, \tilde{X}_t), \nabla^t \rho_t(X_t, \cdot)(\tilde{X}_t) \right\rangle_t \mathbf{1}_{\{X_t \neq \tilde{X}_t\}} \Big\} \mathrm{d}t, \end{aligned} \tag{4.3}$$ 3e6

holds for some one-dimensional Brownian motion b_t.

4.1 The gradient estimates

By using the coupling with parallel displacement, the following gradient estimate is direct. See e.g. [26, 30] for the case with constant metric and see also [12] for the inhomogeneous case without boundary.

2t3 **Theorem 4.2.** *Assume*

$$\mathcal{R}_t^Z \ge K(t), \text{ for some } K \in C([0, T_c)) \text{ and } \mathbb{I}_t \ge 0, \ t \in [0, T_c) \tag{4.4}$$ CV1

holds, then

$$|\nabla^s P_{s,t} f|_s \le e^{-\int_s^t K(r)\mathrm{d}r} P_{s,t} |\nabla^t f|_t, \quad f \in C_b^1(M), \ 0 \le s \le t < T_c.$$

If $(\partial M, g_t)$ keep convex, i.e. $\mathbb{I}_t \ge 0, \ t \in [0, T_c)$, we call the metric flow is convex flow. If it is not a convex flow, combining Theorem 4.2 with the conformal change of metric to make the boundary from concave to convex, we have the following result. Since for $d = 1$ a connected manifold with boundary must be an interval, which is thus convex, in what followings, we only consider $d \ge 2$. Let

$$\mathscr{D} = \{\phi \in C^{1,\infty}([0, T_c) \times M) : \inf \phi_t = 1, \ \mathbb{I}_t \ge -N_t \log \phi_t\}. \tag{4.5}$$ D

Then, by [33, Lemma 2.1], let $\phi \in \mathscr{D}$, $\tilde{g}_t := \phi_t^2 g_t$ is a convex flow.

P1 **Proposition 4.3.** *Let $d \ge 2$ and*

$$\mathrm{Ric}_t^Z \ge K_1(t), \ \partial_t g_t \le K_2(t) \text{ for some } K_1, K_2 \in C([0, T_c)). \tag{4.6}$$ CV2

If there exists $\phi \in \mathscr{D}$ *such that*

$$\begin{aligned}
K_{\phi,1}(t) &:= \inf_M \{\phi_t^2 K_1(t) + \frac{1}{2} L_t \phi_t^2 - |\nabla^t \phi_t^2|_t \cdot |Z_t|_t - (d-2)|\nabla^t \phi_t|_t^2\} > -\infty; \\
K_{\phi,2}(t) &:= \sup_M \{-2\partial_t \log \phi_t + K_2(t)\} > \infty.
\end{aligned}$$

And define

$$K_\phi(t) := K_{\phi,1}^-(t) + K_{\phi,2}(t) + 2\|\phi_t Z_t + (d-2)\nabla^t \phi_t\|_\infty \|\nabla^t \phi_t\|_\infty + (d-2)\|\nabla^t \phi_t\|_\infty^2,$$

then for any $f \in C_b^1(M)$,

$$|\nabla^s P_{s,t} f|_s \leq \|\phi_t\|_\infty \|\nabla^t f\|_\infty e^{\int_s^t K_\phi(r)\mathrm{d}r}, \quad 0 \leq s \leq t < T_c.$$

Proof. Let $\tilde{\Delta}_t$ and $\tilde{\nabla}^t$ be the Laplacian and gradient operator associated to $\tilde{g}_t = \phi_t^{-2} g_t$. According to [29, (2.2)],

$$\phi_t^2 L_t = \tilde{\nabla}^t + \tilde{Z}_t, \quad \tilde{Z}_t = \phi_t Z_t + \frac{d-2}{2} \nabla^t \phi_t^2.$$

By [7, Theorem 1.129] and [15, (3.2)], for any $X \in TM$ such that $\tilde{g}_t(X,X) = 1$, i.e. $|X|_t = \phi_t$, we have

$$\widetilde{\mathrm{Ric}}_t(X,X) = \mathrm{Ric}_t(X,X) + (d-2)\phi_t^{-1} \mathrm{Hess}_{\phi_t}^t(X,X) + \frac{1}{2}\nabla^t \phi_t^2 - (d-2)|\nabla^t \phi_t|_t^2,$$

and

$$\begin{aligned}
\tilde{g}_t(\tilde{\nabla}_X^t \tilde{Z}_t, X) = &\left\langle \nabla_X^t Z_t, X \right\rangle_t + 2\left\langle \nabla^t \log \phi_t, X \right\rangle_t \langle Z_t, X \rangle_t \\
&+ (d-2)\phi_t^{-1} \mathrm{Hess}_{\phi_t}^t(X,X) + (d-2)\left\langle X, \nabla^t \log \phi_t \right\rangle_t^2 \\
&- \phi_t \left\langle Z_t, \nabla^t \phi_t \right\rangle_t - (d-2)|\nabla^t \phi_t|_t^2.
\end{aligned}$$

Therefore, noting that $|X|_t = \phi_t$,

$$\begin{aligned}
\widetilde{\mathrm{Ric}}_t^{\tilde{Z}}(X,X) &= \widetilde{\mathrm{Ric}}_t(X,X) - \tilde{g}_t(\tilde{\nabla}_X^t \tilde{Z}_t, X) \\
&= \mathrm{Ric}_t^Z(X,X) + \frac{1}{2} L_t \phi_t^2 - 2\left\langle \nabla^t \log \phi_t, X \right\rangle_t \langle Z_t, X \rangle_t - (d-2)\left\langle X, \nabla^t \log \phi_t \right\rangle_t^2 \\
&\geq K_1(t)\phi_t^2 + \frac{1}{2} L_t \phi_t^2 - |\nabla^t \phi_t^2|_t \cdot |Z_t|_t - (d-2)|\nabla^t \phi_t|_t^2 \\
&\geq K_{\phi,1}(t).
\end{aligned}$$

and

$$\begin{aligned}
\partial_t \tilde{g}_t(X,X) = \partial_t(\phi_t^{-2} g_t)(X,X) &= (\partial_t \phi_t^{-2})\phi_t^2 + \phi_t^{-2} \partial_t g_t(X,X) \\
&\geq -\partial_t \log \phi_t + K_2(t) \geq K_{\phi,2}(t).
\end{aligned}$$

Since $L_t = \phi_t^{-2}(\tilde{\Delta}^t + \tilde{Z}_t)$, then $\{P_{s,t}\}_{0 \leq s \leq t < T_c}$ is the inhomogeneous semigroup for the solution to the SDE:

$$\mathrm{d}_I X_t = \phi_t^{-1} u_t \mathrm{d}B_t + \phi_t^{-2} \tilde{Z}_t(X_t)\mathrm{d}t + \tilde{N}_t(X_t)\mathrm{d}l_t,$$

where d_I denote the Itô differential on M. In local coordinates the Itô differential for a continuous semi-martingale X_t on M is given by (see e.g. [13])

$$(\mathrm{d}_I X_t)^k = \mathrm{d}X_t^k + \frac{1}{2}\sum_{i,j=1}^{d} \Gamma_{i,j}^k(t, X_t)\mathrm{d}\left\langle X^i, X^j\right\rangle_t, \quad 1 \le k \le d,$$

where $\Gamma_{ij}^k(t,x)$ are the Christoffel symbols w.r.t. g_t. Let X_t solve above SDE with $X_0 = x$, and let Y_t solve

$$\mathrm{d}_I Y_t = \phi_t^{-1} P_{X_t,Y_t}^t u_t \mathrm{d}B_t + \phi_t^{-2}\tilde{Z}_t(Y_t)\mathrm{d}t + \tilde{N}_t(Y_t)\mathrm{d}\tilde{l}_t,$$

for $Y_0 = y$, where $\tilde{l}_t$ is the local time of Y_t on ∂M. (X_t, Y_t) is the coupling by parallel displacement of the reflecting L_t-diffusion process. Moreover, ∂M is convex under $\tilde{g}_t$, then by the second variational formula and the index lemma, for $t \in [0, T_c)$,

$$\mathrm{d}\tilde{\rho}_t(X_t, Y_t) \le (\phi_t^{-1}(X_t) - \phi_t^{-1}(Y_t)) \cdot \tilde{g}_t(\tilde{\nabla}^t \tilde{\rho}_t(\cdot, Y_t)(X_t), u_t \mathrm{d}B_t) + K_\phi(t)\tilde{\rho}_t(X_t, Y_t)\mathrm{d}t,$$

where $K_\phi(t) = K_{\phi,1}^-(t) + 2\|\phi_t Z_t + (d-2)\nabla^t\phi_t\|_\infty \|\nabla^t \phi_t\|_\infty + (d-1)\|\nabla^t\phi_t\|_\infty^2 + K_{\phi,2}(t)$. In addition, $\phi_t \ge 1$, we therefore have $\tilde{\rho}_t \le \rho_t \le \|\phi_t\|_\infty \tilde{\rho}_t$, which implies

$$\rho_t(X_t, Y_t) \le \|\phi_t\|_\infty e^{\int_0^t K_\phi(s)\mathrm{d}s}\tilde{\rho}_0(x,y), \ 0 \le t < T_c.$$

Because

$$\begin{aligned}\left|\frac{P_t f(x) - P_t f(y)}{\rho_0(x,y)}\right| &= \left|\mathbb{E}\left[\frac{f(X_t) - f(Y_t)}{\rho_t(X_t, Y_t)} \frac{\rho_t(X_t, Y_t)}{\rho_0(x,y)}\right]\right| \\ &\le \|\nabla^t f\|_\infty \|\phi_t\|_\infty e^{\int_0^t K_\phi(s)\mathrm{d}s},\end{aligned}$$

we obtain the result directly.

□

c2 **Corollary 4.4.** *Assume (4.6) holds, if there exists $\phi \in \mathscr{D}$ such that $K_\phi(t) < \infty$ for $0 \le s < t < T_c$, then*

$$|\nabla^s P_{s,t} f|_s \le \|\phi_t\|_\infty (P_{s,t}|\nabla^t f|_t^{p/(p-1)})^{(p-1)/p} e^{-\int_s^t (K_1 - \frac{1}{2}K_2 + K_\phi^{(p)})(u)\mathrm{d}u}, \quad f \in C_b^1(M)$$

holds for $p \in [1, \infty)$ and $K_\phi^{(p)}(r) := \inf\{\phi_r^{-1}(L_r\phi_r) + \partial_r \log\phi_r - (p+1)|\nabla^r \log \phi_r|_r^2\}$. Moreover,

$$|\nabla^s P_{s,t} f|_s^2 \le \frac{1}{2}\left[\int_s^t \|\phi_u\|_\infty^{-2} e^{2\int_s^u (K_1 - \frac{1}{2}K_2 + K_\phi^{(2)})(r)\mathrm{d}r}\mathrm{d}u\right]^{-1} P_{s,t} f^2, \quad t \in [0, T_c), \ f \in \mathscr{B}_b(M). \quad (4.7)$$ eq1

Proof. Without loss generality, we consider $s = 0$ for simplicity.
(a) By the Itô formula, we have

$$\begin{aligned}\mathrm{d}\phi_t^{-p}(X_t) &= \left\langle \nabla^t \phi_t^{-p}(X_t), u_t \mathrm{d}B_t\right\rangle_t + (L_t \phi_t^{-p}(X_t) + \partial_t \phi_t^{-p}(X_t))\mathrm{d}t + N_t \phi_t^{-p}(X_t)\mathrm{d}l_t \\ &\le \left\langle \nabla^t \phi_t^{-p}(X_t), u_t \mathrm{d}B_t\right\rangle_t - p\phi_t^{-p}(X_t)\{K_\phi^{(p)}(t)\mathrm{d}t + N_t \log\phi_t(X_t)\mathrm{d}l_t\}.\end{aligned}$$

So $M_t := \phi_t^{-p}(X_t)\exp\left[p\int_0^t K_\phi^{(p)}(s)\mathrm{d}s + p\int_0^t N_s\log\phi_s(X_s)\mathrm{d}l_s\right]$ is a local martingale. Thus, by the Fatou lemma, and noting that $\phi_t \geq 1$,

$$\begin{aligned}
&\mathbb{E}\left\{\phi_t^{-p}(X_t)\exp\left[p\int_0^t K_\phi^{(p)}(s)\mathrm{d}s + p\int_0^t N_s\log\phi_s(X_s)\mathrm{d}l_s\right]\right\}\\
&\leq \liminf_{n\to\infty}\mathbb{E}^x\left\{\phi_t^{-p}(X_{t\wedge\zeta_n})\exp\left[p\int_0^{t\wedge\zeta_n} K_\phi^{(p)}(s)\mathrm{d}s + p\int_0^{t\wedge\zeta_n} N_s\log\phi_s(X_s)\mathrm{d}l_s\right]\right\}\\
&\leq \phi_0^{-p}(x) \leq 1.
\end{aligned}$$

Therefore,

$$\mathbb{E}^x\exp\left[p\int_0^t N_s\log\phi_s(X_s)\mathrm{d}l_s\right] \leq \|\phi_t\|_\infty^p e^{-p\int_0^t K_\phi^p(s)\mathrm{d}s}, \quad t\geq 0. \tag{4.8}$$

Since $\mathbb{I}_t \geq -N_t\log\phi_t$, by combining this with Theorem 3.1 for $\sigma(t,\cdot) = -N_t\log\phi_t$ and Proposition 4.3, we obtain

$$\begin{aligned}
&|\nabla^0 P_{0,t}f|_0^p(x) \leq (P_{0,t}|\nabla^t f|_t^{p/(p-1)})^{(p-1)}(x)\mathbb{E}^x\|Q_t\|^p\\
\leq&(P_{0,t}|\nabla^t f|_t^{p/(p-1)}(x))^{(p-1)}\mathbb{E}^x\exp\left[-p\int_0^t\left(K_1(s)-\frac{1}{2}K_2(s)\right)\mathrm{d}s + p\int_0^t N_s\log\phi_s(X_s)\mathrm{d}s\right]\\
\leq&\|\phi_t\|_\infty^p(P_{0,t}|\nabla^t f|_t^{p/(p-1)})^{(p-1)}(x)e^{-p\int_0^t(K_1-\frac{1}{2}K_2+K_\phi)(s)\mathrm{d}s}.
\end{aligned}$$

Therefore, the first inequality holds.

(b) Take

$$h(s) = \frac{\int_0^s\|\phi_u\|_\infty^{-2}e^{2\int_0^u(K_1-\frac{1}{2}K_2+K_\phi^{(2)})(r)\mathrm{d}r}\mathrm{d}u}{\int_0^t\|\phi_u\|_\infty^{-2}e^{2\int_0^u(K_1-\frac{1}{2}K_2+K_\phi^{(2)})(r)\mathrm{d}r}\mathrm{d}u}, \quad s\in[0,t].$$

Then the following inequality follows from the second formula in (3.2) and (4.8) for $p=2$,

$$\begin{aligned}
|\nabla^0 P_{0,t}f|_0^2 &\leq \frac{P_{0,t}f^2}{2}\mathbb{E}\int_0^t h'(s)^2\|Q_s\|^2\mathrm{d}s\\
&\leq \frac{P_{0,t}f^2}{2}\mathbb{E}\int_0^t h'(s)^2\|\phi_s\|_\infty^2\exp\left[-2\int_0^s\left(K_1-\frac{1}{2}K_2+K_\phi^{(2)}\right)(r)\mathrm{d}r\right]\mathrm{d}s\\
&\leq \frac{1}{2}\left[\int_0^t\|\phi_u\|_\infty^{-2}e^{2\int_0^u(K_1-\frac{1}{2}K_2+K_\phi^{(2)})(r)\mathrm{d}r}\mathrm{d}u\right]^{-1}P_{0,t}f^2.
\end{aligned}$$

□

Recall that $\rho_t^\partial(x)$ is the distance between x and ∂M w.r.t. g_t. Let Sect_t be the section curvature of M under g_t. We want to present an explicit construction of $\phi\in C_b^{1,2}([0,T_c)\times M)$ under the following assumption, which is trivial when M is compact.

(A1) At least one of the following holds: for $t\in[0,T_c)$,

(i) ∂M is convex under g_t;

(ii) $\mathbb{I}_t$ is bounded, $\partial_t g_t$ is bounded above and there exists $r_0>0$ such that on the set $\partial_{r_0}^t M := \{x\in M : \rho_t^\partial(x)\leq r_0\}$, ρ_t^∂ is smooth, $\partial_t g_t$ and Z_t are bounded, and Sect_t is bounded above.

Under the assumption (**A1**), we will be able to construct the desired function ϕ_t by ρ_t^∂. Thus, as to calculate K_ϕ and $K_\phi^{(p)}$, we shall make use of the Laplacian comparison theorem. For any $\theta, k, \sigma \in \mathbb{R}^+$, let

$$h(s) = \cos(\sqrt{k}\, s) - \frac{\theta}{\sqrt{k}} \sin(\sqrt{k}\, s), \quad s \geq 0. \tag{4.9}$$ h

Then $h^{-1}(0) = k^{-1/2} \arcsin \frac{\sqrt{k}}{\sqrt{k+\theta^2}}$. Moreover, let

$$\delta = \delta(r_0, \sigma, k, \theta) = \frac{\sigma(1 - h(r_0))^{d-1}}{\int_0^{r_0} (h(s) - h(r_0))^{d-1} \mathrm{d}s}. \tag{4.10}$$ delta

2t4 **Theorem 4.5.** *Assume* (**A1**) *holds and for any* $[s,t] \subset [0, T_c)$, *let* $K \in C_b([s,t] \times M)$ *and* $\sigma \in C_b([s,t] \times \partial M)$ *such that*

$$\mathcal{R}_Z^t \geq K(t, \cdot) \text{ and } \mathbb{I}_t \geq \sigma(t, \cdot), \text{ for } t \in [0, T_c), \tag{4.11}$$ CV3

(1) *there exists a progressively measurable process* $\{Q_{s,t}\}_{0 \leq s \leq t < T_c}$ *on* $\mathbb{R}^d \times \mathbb{R}^d$ *such that*

$$Q_{s,s} = I, \quad \|Q_{s,t}\| \leq \exp\left[-\int_s^t K(r, X_r)\mathrm{d}r - \int_s^t \sigma(r, X_r)\mathrm{d}l_r\right],$$

and for any $h \in C^1([s,t])$ *such that* $h(s) = 0,\ h(t) = 1$, *and for any* $f \in C_b^1(M)$,

$$\begin{aligned} u_s^{-1} \nabla^s P_{s,t} f(x) &= \mathbb{E}\{Q_{s,t}^* u_t^{-1} \nabla^t f(X_t) | X_s = x\} \\ &= \mathbb{E}\left\{\frac{f(X_t)}{\sqrt{2}} \int_s^t h'(r) Q_{s,r}^* \mathrm{d}B_r | X_s = x\right\}. \end{aligned} \tag{4.12}$$

(2) *let* K *and* σ *belong to* $C^1([0, T_c))$, *and let* $\mathbb{I}_s \leq \theta(s)$, $\mathrm{Sect}_s|_{\partial_{r_0}^s M} \leq k(s)$ *hold for all* $s \in [0, T_c)$. *Let* $\theta_{[0,t]} = \sup_{s \in [0,t]} \theta(s)$, $k_{[0,t]} = \sup_{s \in [0,t]} k(s)$, $\sigma_{[0,t]} = \sup_{s \in [0,t]} \sigma(s)$. *Then* (4.7) *and*

$$\mathbb{E}^x \exp\left[-p \int_0^t \sigma(s) \mathrm{d}l_s\right] \leq \|\phi_t\|_\infty^p e^{-p \int_0^t K_\phi^{(p)}(s)\mathrm{d}s}, \quad x \in M,\ t \geq 0 \tag{4.13}$$

hold for $K_\phi^{(p)} = 0$ *and* $\phi = 1$ *if* $\mathbb{I}_s \geq 0$ *for* $t \in [0, T_c)$, *and for*

$$K_\phi^{(p)}(s) = -\delta_{[0,t]} - \sigma_{[0,t]}^- \sup_{\partial_{r_0} M} (|Z_s|_s + r_0 \partial_s g_s) - (p+1)(\sigma_{[0,t]}^-)^2$$

and

$$\|\phi_s\|_\infty = 1 + \delta_{[0,t]} \int_0^{r_0} (h(r) - h(r_0))^{1-d} \mathrm{d}r \int_r^{r_0} (h(u) - h(r_0))^{d-1} \mathrm{d}u \leq 1 + \frac{r_0^2}{\delta_{[0,t]}},$$

where h, $\delta_{[0,t]}$ *are defined as in* (4.9) *and* (4.10) *where* σ, k, θ *are replaced by* $\sigma_{[0,t]}, k_{[0,t]}, \theta_{[0,t]}$, *if (ii) in* (**A1**) *holds with* $r_0 \leq k_{[0,t]}^{-1/2} \arcsin \frac{\sqrt{k_{[0,t]}}}{\sqrt{k_{[0,t]} + \theta_{[0,t]}^2}}$.

Proof. According to Theorem 3.1, it suffices to prove (2). Moreover, by Theorem 4.2, when $\{g_t\}_{t \in [0, T_c)}$ is a convex flow, the desired assertions follow immediately by taking $\phi \equiv 1$. So it remain to prove (2) for non-convex flow case. Let $\phi_s = \varphi \circ \rho_s^\partial$. For simplicity, we will drop the subscript $[0,t]$.

$$\phi(r) = 1 + \delta \int_{r \wedge r_0}^{r_0} (h(s) - h(r_0))^{1-d} \mathrm{d}s \int_{s \wedge r_0}^{r_0} (h(u) - h(r_0))^{d-1} \mathrm{d}u.$$

By a approximation argument we may regard ϕ as C^∞-smooth. Obviously, $\phi \geq 1$, $N_s \log \phi_s = -\sigma \geq \mathbb{I}_s$, $s \in [0,t]$. Since $\phi' \geq 0$, according to the Laplacian comparison theorem for ρ_s^∂ (see [20, 21]), we have

$$\Delta_s \phi_s^\partial \geq \left(\frac{(d-1)\varphi' h'}{h} + \varphi'' \right) (\rho_s^\partial) \geq \delta, \ s \in [0,t].$$

Since $|\nabla^s \log \phi_s|_s$, $|Z_s|_s$, $\partial_s g_s$ are bounded on $\partial_{r_0} M$, this implies that $K_\phi(s) > -\infty$. Noting that $\varphi'' \geq 0$, we have

$$\begin{aligned}\frac{1}{\phi_s} L_s \phi_s + \partial_s \log \phi_s - \frac{p+1}{\phi_s^2} |\nabla^s \phi_s|_s^2 &\geq -\delta + \varphi'(0) \sup_{\partial_{r_0} M} (|Z_s|_s + \partial_s g_s r_0) - (p+1)\varphi'(0)^2 \\ &= -\delta - \sigma \sup_{\partial_{r_0} M} (|Z_s|_s + \partial_s g_s r_0) - (p+1)\sigma^2.\end{aligned}$$

Therefore we complete proof by combining this with (4.7) and (4.8). $\square$

5 Equivalent inequalities for curvature conditions

In this section, we present the gradient estimates for the curvature conditions, i.e. the lower bounds of $\mathcal{R}_t^Z$ and $\mathbb{I}_t$, which is an expansion of [12, Theorem 4.3] for the inhomogeneous manifold without boundary. This part is mainly based on [35, Theorem 1.1] for the case when the metric is independent of t.

4t2 **Theorem 5.1.** *Assume $p \in [1,\infty)$, $\tilde{p} = p \wedge 2$. Then for any $[s,t] \subset [0,T_c)$, $K \in C_b([s,t] \times M)$ and $\sigma \in C_b([s,t] \times \partial M)$, the following statements are equivalent to each other:*

(1) (4.11) *holds. for any $0 \leq t < T_c$.*

(2) $|\nabla^s P_{s,t} f(x)|_s^p \leq \mathbb{E}\{|\nabla^t f|_t^p(X_t) \exp[-p\int_s^t K(u,X_u)\mathrm{d}u - p\int_s^t \sigma(u,X_u)\mathrm{d}l_u] | X_s = x\}$ *holds for $x \in M$, and $0 \leq s \leq t < T_c$, $f \in C_b^1(M)$.*

(3) *For any $0 \leq s \leq t < T_c$, $x \in M$ and positive $f \in C_b^1(M)$,*
$$\frac{\tilde{p}[P_{s,t} f^2 - (P_{s,t} f^{1/\tilde{p}})^{\tilde{p}}]}{4(\tilde{p}-1)} \leq \mathbb{E}\left\{ |\nabla^t f|_t^2(X_t) \int_s^t e^{-2\int_u^t K(r,X_r)\mathrm{d}r - 2\int_u^t \sigma(r,X_r)\mathrm{d}l_r} \mathrm{d}u \big| X_s = x \right\},$$
where when $p=1$, the inequality is understood as its limit as $p \downarrow 1$:
$$\begin{aligned}&P_{s,t}(f^2 \log f^2)(x) - (P_{s,t} f^2 \log P_{s,t} f^2)(x) \\ &\leq 4\mathbb{E}\left\{ |\nabla^t f|_t(X_t) \int_s^t e^{-2\int_u^t K(r,X_r)\mathrm{d}r - 2\int_u^t \sigma(r,X_r)\mathrm{d}l_r} \mathrm{d}u \big| X_s = x \right\}.\end{aligned}$$

(4) *For any $0 \leq s < t < T_c$, $x \in M$ and positive $f \in C_b^1(M)$,*
$$\begin{aligned}&|\nabla^s P_{s,t} f|_s^2(x) \\ &\leq \frac{[P_{s,t} f^{\tilde{p}} - (P_{s,t} f)^{\tilde{p}}](x)}{\tilde{p}(\tilde{p}-1)\int_s^t \left(\mathbb{E}\{(P_{u,t} f)^{2-\tilde{p}}(X_u) \exp\left[-2\int_s^u K(r,X_r)\mathrm{d}r - 2\int_s^u \sigma(r,X_r)\mathrm{d}l_r\right] | X_s = x\} \right)^{-1} \mathrm{d}u},\end{aligned}$$
where when $p=1$, the inequality is understood as its limit as $p \downarrow 1$:
$$|\nabla^s P_{s,t} f|_s^2(x) \leq \frac{[P_{s,t}(f \log f) - (P_{s,t} f) \log P_{s,t} f](x)}{\displaystyle\int_s^t \left(\mathbb{E}\left\{ P_{u,t} f(X_u) e^{-2\int_s^u K(r,X_r)\mathrm{d}r - 2\int_s^u \sigma(r,X_r)\mathrm{d}l_r} \big| X_s = x \right\} \right)^{-1} \mathrm{d}u}.$$

Proof. By the derivative formula established in Theorem 3.1, it is easy to derive (2) from (1); then according to Theorem 3.3, we see that (2)–(4) implies (1); and finally, taking $f \in C_c^\infty(M)$, $N_t f = 0$ and f is constant outside a compact set, we derive (3), (4) from (2) by a similarly discussion as in [39, the proof of Theorem 2.3.1] for the case with constant metric or [12, Theorem 4.1] for the extensions to time-inhomogeneous manifolds without boundary. $\square$

r1 **Remark 5.2.** *When the family of the metric evolves under Ricci flow with boundary condition $\mathbb{I}_t = \lambda \langle \cdot, \cdot \rangle_t$, $\lambda \in \mathbb{R}$ (see [27] for the short time existence of the flow). Then for the g_t- Brownian motion, we have*

$$|\nabla^s P_{s,t} f(x)|_s^p \le \mathbb{E}\{|\nabla^t f|_t^p(X_t) \exp[-p\lambda(l_t - l_s)] | X_s = x\}$$

holds for $x \in M$, and $0 \le s \le t < T_c$, $f \in C_b^1(M)$.

Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing function, we define a cost function

$$C_t(x, y) = \varphi(\rho_t(x, y)).$$

To the cost function C_t, we associate the Monge-Kantorovich minimization between two probability measures on M,

$$W_{C_t}(\mu, \nu) = \inf_{\eta \in \mathscr{C}(\mu,\nu)} \int_{M\times M} C_t(x, y) \mathrm{d}\eta(x, y), \tag{5.1}$$ eq3

where $\mathscr{C}(\mu, \nu)$ is the set of all probability measures on $M \times M$ with marginal μ and ν. We denote

$$W_{p,t}(\mu, \nu) = (W_{\rho_t^p}(\mu, \nu))^{1/p}$$

the Wasserstein distance associated to $p > 0$.

Below this is our main result in this section, which is an extension of [12, Theorem 4.3] to manifolds carrying convex flow. From technic viewpoint. See [34, 36, 37] the corresponding conclusions for the constant manifold with boundary.

4t3 **Theorem 5.3.** *Let $p \in [1, \infty)$, and let $\{p_{s,t}\}_{0 \le s \le t < T_c}$ be the heat kernel of $\{P_{s,t}\}_{0\le s\le t<T_c}$ w.r.t. measure μ_t equivalent to the volume measure w.r.t. g_t. Then the following assertions are equivalent each other:*

(1) *(4.4) holds*

(2) *For any $x, y \in M$ and $0 \le s < t < T_c$,*

$$W_{p,t}(\delta_x P_{s,t}, \delta_y P_{s,t}) \le \rho_s(x, y) e^{-\int_s^t K(r)\mathrm{d}r}.$$

(2′) *For any $\mu_1, \mu_2 \in \mathscr{P}(M)$, the space of all probability measure on M, and $0 \le s < t < T_c$,*

$$W_{p,t}(\nu_1 P_{s,t}, \nu_2 P_{s,t}) \le W_{p,s}(\nu_1, \nu_2) e^{-\int_s^t K(r)\mathrm{d}r}.$$

(3) *When $p>1$, for any $f\in\mathscr{B}_b^+(M)$ and $0\le s<t<T_c$,*

$$(P_{s,t}f)^p(x)\le P_{s,t}f^p(y)\exp\left[\frac{p}{p-1}C(s,t,K)\rho_s^2(x,y)\right],$$

where $C(s,t,K)=\left[4\int_s^t e^{2\int_s^r K(u)\mathrm{d}u}\mathrm{d}r\right]^{-1}$. It keeps the same meaning in (4)–(6).

(4) *For any $f\in\mathscr{B}_b^+(M)$ with $f\ge 1$ and $0\le s\le t<T_c$,*

$$P_{s,t}\log f(x)\le \log P_{s,t}f(y)+C(s,t,K)\rho_s^2(x,y).$$

(5) *When $p>1$, for any $0\le s\le t<T_c$ and $x,y\in M$,*

$$\int_M p_{s,t}(x,y)\left(\frac{p_{s,t}(x,y)}{p_{s,t}(y,z)}\right)^{\frac{1}{p-1}}\mu_t(\mathrm{d}z)\le \exp\left[\frac{p}{(p-1)^2}C(s,t,K)\rho_s^2(x,y)\right].$$

(6) *For any $0\le s<t<T_c$ and $x,y\in M$,*

$$\int_M p_{s,t}(x,y)\log\frac{p_{s,t}(x,y)}{p_{s,t}(y,z)}\mu_t(\mathrm{d}z)\le \rho_s^2(x,y)C(s,t,K).$$

(7) *For any $0\le s<u\le t<T_c$ and $1<q_1\le q_2$ such that*

$$\frac{q_2-1}{q_1-1}=\frac{\int_s^t e^{2\int_s^z K(r)\mathrm{d}r}\mathrm{d}z}{\int_s^u e^{2\int_s^z K(r)\mathrm{d}r}\mathrm{d}z}\tag{5.2}$$

there holds

$$\{P_{s,u}(P_{u,t}f)^{q_2}\}^{\frac{1}{q_2}}\le (P_{s,t}f^{q_1})^{\frac{1}{q_1}},\ f\in\mathscr{B}_b^+(M).$$

(8) *For any $0\le s\le u\le t<T_c$ and $0<q_2\le q_1$ or $q_2\le q_1<0$ such that (5.2) holds,*

$$(P_{s,t}f^{q_1})^{\frac{1}{q_1}}\le \{P_{s,u}(P_{u,t}f)^{q_2}\}^{\frac{1}{q_2}},\quad f\in\mathscr{B}_b^+(M).$$

(9) *For any $0\le s<u\le t<T_c$ and $f\in C_b^1(M)$,*

$$|\nabla^s P_{s,t}f|_s^p\le e^{-p\int_s^t K(r)\mathrm{d}r}P_{s,t}|\nabla^t f|_t^p.$$

(10) *For any $0\le s<u\le t<T_c$ and positive $f\in C_b^1(M)$,*

$$\frac{(p\wedge 2)\{P_{s,t}f^2-(P_{s,t}f^{2/(p\wedge 2)})^{p\wedge 2}\}}{4(p\wedge 2-1)}\le\left(\int_s^t e^{-2\int_u^t K(r)\mathrm{d}r}\mathrm{d}u\right)P_{s,t}|\nabla^t f|_t^2.$$

When $p=1$, the inequality reduces to the log-Sobolev inequality

$$P_{s,t}(f^2\log f^2)-(P_{s,t}f^2)\log P_{s,t}f^2\le\left(4\int_s^t e^{-2\int_u^t K(r)\mathrm{d}r}\mathrm{d}u\right)P_{s,t}|\nabla^t f|_t^2.$$

Proof. First, by Theorems 2.3, 4.1 and 5.1, the inequalities (2)– (10) can be derived from (1) by a similar discussion as in [12, Theorem 4.4] for the case without boundary.

Then, we assume (4) and prove (1). For a fixed point $x\in M^\circ$, $t\in[0,T_c)$ and $X\in T_xM$, taking $f\in C_0^\infty(M)$ such that $\nabla^t f=X$, $\mathrm{Hess}_f^t(x)=0$ and $f=0$ in a neighborhood of ∂M. The argument in [12, Theorem 4.4] works well for this case, i.e. $\mathcal{R}_t^Z\ge K(t)$ can be induced from (4).

So, it only leaves for us to derive $\mathbb{I}_t\ge 0$. Since by Theorem 3.4, we only need to consider the term with order $\sqrt{t}$. So we do not need to care about the terms, which come from $\partial_t g_t$, since they at least have order t. Therefore, by a similar procedure as in time-homogeneous case (see [35]). We conclude that ∂M is convex under g_t for all $t\in[0,T_c)$. $\square$

6 Coupling method for Harnack inequality and extension to non-convex flow

In this section, we want to use coupling method to give an alternative proof of Harnack inequality, i.e. "(1) implies (3), (4) in Theorem 5.3".

6.1 Harnack inequality by coupling method

In this section, we suppose (4.4) holds. Let $x, y \in M$, $T \in (0, T_c)$ and $p > 1$ be fixed such that $x \neq y$. For $\theta \in (0,2)$, let

$$\xi_t = (2-\theta)e^{-2\int_0^t K(r)\mathrm{d}r}\left(\int_0^T e^{2\int_0^s K(r)\mathrm{d}r}\mathrm{d}s - \int_0^t e^{2\int_0^s K(r)\mathrm{d}r}\mathrm{d}s\right). \tag{6.1}$$

Then ξ is smooth and strictly positive on $[0,T]$ such that

$$2 + 2K(t)\xi_t + \xi_t' = \theta, \quad t \in [0, T_c), \tag{6.2}$$

and

$$\xi_0 = (2-\theta)\int_0^T e^{2\int_0^s K(r)\mathrm{d}r}\mathrm{d}s.$$

Consider the coupling

$$\begin{cases} \mathrm{d}_I X_t = \sqrt{2}u_t \mathrm{d}B_t + Z_t(X_t)\mathrm{d}t + N_t(X_t)\mathrm{d}l_t, & X_0 = x, \\ \mathrm{d}_I Y_t = \sqrt{2}P^t_{X_t,Y_t}u_t\mathrm{d}B_t + Z_t(Y_t)\mathrm{d}t + \frac{1}{\xi_t}\rho_t(X_t,Y_t)\nabla^t\rho_t(\cdot,Y_t)(X_t)\mathrm{d}t + N_t(Y_t)\mathrm{d}\tilde{l}_t, & Y_0 = y. \end{cases} \tag{6.3}$$

Since the additional drift term $\xi_t^{-1}\rho_t(x,y)\nabla^t\rho_t(\cdot,y)(x)$ is locally differentiable, the coupling (X_t, Y_t) is a well-defined continuous process for $t < \zeta$, where ζ is the explosion time of Y_t; namely, $\zeta = \lim_{n\to\infty}\zeta_n$ for

$$\zeta_n := \inf\{t \in [0,T] : \rho_t(Y_t, o) \geq n\}, \quad \inf\varnothing = T.$$

Let

$$\mathrm{d}\tilde{B}_t = \mathrm{d}B_t + \frac{\rho_t(X_t,Y_t)}{\sqrt{2}\xi_t}\mathrm{d}t.$$

If $\zeta = T$,

$$R_s := \exp\left[-\int_0^s \frac{\rho_t(X_t,Y_t)}{\sqrt{2}\xi_t}\left\langle\nabla^t\rho_t(\cdot,Y_t)(X_t), u_t\mathrm{d}B_t\right\rangle_t - \frac{1}{4}\int_0^t \frac{1}{\xi_t^2}\rho_t(X_t,Y_t)^2\mathrm{d}t\right]$$

is a uniformly integrable martingale for $s \in [0,T)$, then by the martingale convergence theorem, $R_T := \lim_{t\uparrow T} R_t$ exists and $\{R_t\}_{t\in[0,T)}$ is a martingale. In this case, by the Girsanov theorem $\{\tilde{B}_t\}_{t\in[0,T)}$ is a d-dimensional Brownian motion under the probability $R_T\mathbb{P}$. Rewrite (6.3) as

$$\begin{cases} \mathrm{d}_I X_t = \sqrt{2}u_t\mathrm{d}\tilde{B}_t + Z_t(X_t)\mathrm{d}t - \frac{\rho_t(X_t,Y_t)}{\xi_t}\nabla^t\rho_t(\cdot,Y_t)(X_t)\mathrm{d}t + N_t(X_t)\mathrm{d}l_t, & X_0 = x, \\ \mathrm{d}_I Y_t = \sqrt{2}P^t_{X_t,Y_t}u_t\mathrm{d}\tilde{B}_t + Z_t(Y_t)\mathrm{d}t + N_t(Y_t)\mathrm{d}\tilde{l}_t, & Y_0 = y. \end{cases} \tag{6.4}$$

By the second variational formula and the index lemma, we have

$$\mathrm{d}\rho_t(X_t,Y_t) \leq -K(t)\rho_t(X_t,Y_t)\mathrm{d}t - \frac{\rho_t(X_t,Y_t)}{\xi_t}\mathrm{d}t, \ t < T.$$

This implies

$$\mathrm{d}\frac{\rho_t(X_t,Y_t)^2}{\xi_t} \le -\frac{\rho_t(X_t,Y_t)^2}{\xi_t^2}(\xi_t' + 2K(t)\xi_t + 2)\mathrm{d}t.$$

In particular, let ξ_t has the form as (6.1). We have $\xi_t' + 2K(t)\xi_t + 2 = \theta$, $t \in [0,T)$. Since $\int_0^T \xi_t^{-1}\mathrm{d}t = \infty$, we will see that the additional drift $-\frac{\rho_t(X_t,Y_t)}{\xi_t}\nabla^t\rho_t(\cdot,Y_t)(X_t)\mathrm{d}t$ is strong enough to force the coupling to be successful up to time T. So we first prove the uniform integrability of $\{R_{s\wedge\zeta}\}_{s\in[0,T)}$ w.r.t. $\mathbb{P}$ so that $R_{T\wedge\zeta} := \lim_{s\uparrow T} R_{s\wedge\zeta}$ exists. Then, prove that $\zeta = T$, $\mathbb{Q}$-a.s. for $\mathbb{Q} := R_{T\wedge\zeta}\mathbb{P}$ so that $\mathbb{Q} = R_T\mathbb{P}$. Let

$$\tau_n = \inf\{t \in [0,T) : \rho_t(o,X_t) + \rho_t(o,Y_t) \ge n\}.$$

Since X_t is non-explosive as assumed, we have $\tau_n \uparrow \zeta$ as $n \uparrow \infty$.

lem1 **Lemma 6.1.** *Assume (4.4). Let $\theta \in (0,2)$, $x, y \in \mathbb{R}^d$ and $T > 0$ be fixed.*

(1) *There holds*

$$\sup_{s\in[0,T),n\ge1} \mathbb{E}R_{s\wedge\tau_n}\log R_{s\wedge\tau_n} \le \frac{\rho_0^2(x,y)}{4\theta(2-\theta)\int_0^T e^{2\int_0^s K(r)\mathrm{d}r}\mathrm{d}s}.$$

Consequently,

$$R_{s\wedge\zeta} := \lim_{n\uparrow\infty} R_{s\wedge\tau_n\wedge(T-\frac{1}{n})}, s \in [0,T_c), \quad R_{T\wedge\zeta} := \lim_{s\uparrow T} R_{s\wedge\zeta}.$$

exist such that $\{R_{s\wedge\zeta}\}_{s\in[0,T]}$ is a uniformly integrable martingale.

(2) *Let $\mathbb{Q} = R_{T\wedge\zeta}\mathbb{P}$. Then $\mathbb{Q}(\zeta = T) = 1$ so that $\mathbb{Q} = R_T\mathbb{P}$.*

Proof. (1) Let $s \in [0,T)$ be fixed. By (6.3) and the curvature condition (4.4), and the Itô formula,

$$\mathrm{d}\rho_t^2(X_t,Y_t) \le -2K(t)\rho_t^2(X_t,Y_t)\mathrm{d}t - 2\frac{\rho_t^2(X_t,Y_t)}{\xi_t}\mathrm{d}t$$

holds for $t \le s\wedge\tau_n$. Combining this with (6.2), we obtain

$$\mathrm{d}\frac{\rho_t^2(X_t,Y_t)}{\xi_t} \le -\frac{\rho_t^2(X_t,Y_t)}{\xi_t^2}\left(\xi_t' + 2K(t)\xi_t + 2\right)\mathrm{d}t = -\frac{\rho_t^2(X_t,Y_t)}{\xi_t^2}\theta\mathrm{d}t. \tag{6.5}$$ e7

Multiplying by $\frac{1}{\theta}$ and integrating from 0 to $s\wedge\tau_n$, we obtain

$$\int_0^{s\wedge\tau_n} \frac{\rho_t^2(X_t,Y_t)}{\xi_t^2}\mathrm{d}t \le -\frac{\rho^2_{g(s\wedge\tau_n)}(X_{s\wedge\tau_n},Y_{s\wedge\tau_n})}{\theta\xi_{s\wedge\tau_n}} + \frac{\rho_0^2(x,y)}{\theta\xi_0}.$$

By the Girsanov theorem, $\{\tilde{B}_t\}_{t\le\tau_n\wedge s}$ is the d-dimensional Brownian motion under the probability measure $R_{s\wedge\tau_n}\mathbb{P}$, so, taking expectation $\mathbb{E}_{s,n}$ with respect to $R_{s\wedge\tau_n}\mathbb{P}$, we obtain

$$\mathbb{E}_{s,n}\int_0^{s\wedge\tau_n} \frac{\rho_t^2(X_t,Y_t)}{\xi_t^2}\mathrm{d}t \le \frac{\rho_0^2(x,y)}{\theta\xi_0}. \tag{6.6}$$ 2eq5

By the definition of R_t and $\tilde{B}_t$, we arrive at

$$\begin{aligned}\log R_r &= -\int_0^r \frac{\rho_t(X_t,Y_t)}{\sqrt{2}\xi_t}\left\langle\nabla^t\rho_t(\cdot,Y_t)(X_t), u_t\mathrm{d}B_t\right\rangle_t - \frac{1}{4}\int_0^r \frac{\rho_t^2(X_t,Y_t)}{\xi_t^2}\mathrm{d}t\\ &= -\int_0^r \frac{\rho_t(X_t,Y_t)}{\sqrt{2}\xi_t}\left\langle\nabla^t\rho_t(\cdot,Y_t)(X_t), u_t\mathrm{d}\tilde{B}_t\right\rangle_t + \frac{1}{4}\int_0^r \frac{\rho_t^2(X_t,Y_t)}{\xi_t^2}\mathrm{d}t.\end{aligned}$$

Since $\{\tilde{B}_t\}$ is the d-dimensional Brownian motion under $R_{s\wedge\tau_n}\mathbb{P}$ up to $s\wedge\tau_n$, combining this with (6.6), we obtain

$$\mathbb{E}R_{s\wedge\tau_n}\log R_{s\wedge\tau_n}=\mathbb{E}_{s,n}\log R_{s\wedge\tau_n}\le\frac{\rho_0^2(x,y)}{4\theta\xi_0}=\frac{\rho_0^2(x,y)}{4\theta(2-\theta)\int_0^T e^{2\int_0^s K(r)\mathrm{d}r}\mathrm{d}s}.$$

By the martingale convergence theorem and the Fatou lemma, $\{R_{t\wedge\xi}: s\in[0,T_c)\}$ is well-defined martingale with

$$\mathbb{E}R_{s\wedge\zeta}\log R_{s\wedge\zeta}\le\frac{\rho_0^2(x,y)}{4\theta(2-\theta)\int_0^T e^{2\int_0^s K(r)\mathrm{d}r}\mathrm{d}s}.$$

To see that $\{R_{s\wedge\zeta}: s\in[0,T]\}$ is a martingale, let $0\le s<t\le T$. By the dominated convergence theorem and the martingale property of $\{R_{s\wedge\tau_n}: s\in[0,T)\}$, we have

$$\begin{aligned}\mathbb{E}(R_{t\wedge\zeta}|\mathscr{F}_s)&=\mathbb{E}(\lim_{n\to\infty}R_{t\wedge\tau_n\wedge(T-1/n)}|\mathscr{F}_s)=\lim_{n\to\infty}\mathbb{E}(R_{t\wedge\tau_n\wedge(T-1/n)}|\mathscr{F}_s)\\&=\lim_{n\to\infty}R_{s\wedge\tau_n}=R_{s\wedge\zeta}.\end{aligned}$$

(2) Let $\sigma_n=\inf\{t\in[0,T]:\rho_t(o,X_t)\ge n\}$. We have $\sigma_n\uparrow\infty$ $\mathbb{P}$-a.s and hence, also $\mathbb{Q}$-a.s. Since $\{\tilde{B}_t\}$ is a $\mathbb{Q}$-Brownian motion up to $T\wedge\zeta$, it follows from (6.5) that there exists a constant $C:=\inf_{t\in[0,T]}e^{2\int_0^t K(r)\mathrm{d}r}>0$ such that

$$\frac{C(n-m)^2}{\xi_0}\mathbb{Q}(\sigma_m>t,\zeta_n\le t)\le\mathbb{E}_{\mathbb{Q}}\frac{\rho^2_{t\wedge\sigma_m\wedge\zeta_n}(X_{t\wedge\sigma_m\wedge\zeta_n},Y_{t\wedge\sigma_m\wedge\zeta_n})}{\xi_{t\wedge\sigma_m\wedge\zeta_n}}\le\frac{\rho_0^2(x,y)}{\xi_0}$$

holds for all $n>m>0$ and $t\in[0,T)$. By letting first $n\uparrow\infty$ then $m\uparrow\infty$, we obtain $\mathbb{Q}(\zeta\le t)=0$ for all $t\in[0,T)$. This is equivalent to $\mathbb{Q}(\zeta=T)=1$ according to the definition of ζ. □

Lemma 6.1 ensures that under $\mathbb{Q}:=R_{T\wedge\zeta}\mathbb{P}$, $\{\tilde{B}_t\}_{t\in[0,T_c)}$ is a Brownian motion. Then by (6.4), the coupling (X_t,Y_t) is well-constructed under $\mathbb{Q}$ for $t\in[0,T]$. Since $\int_0^T\xi_t^{-1}\mathrm{d}t=\infty$, we shall see that the coupling is successful up to time T, so that $X_T=Y_T$ holds $\mathbb{Q}$-a.s.. This will lead to prove the desired Harnack inequality for $\{P_{S,T}\}_{0\le S\le T<T_c}$ according to Theorem 6.3 provided $R_{T\wedge\zeta}$ has finite $p/(p-1)$-moment. The next lemma provides an explicit upper bound on moments of $R_{T\wedge\zeta}$.

lem2 **Lemma 6.2.** *Assume (4.4) holds. Let R_t and ξ_t be fixed for $\theta\in(0,2)$. One has*

$$\sup_{s\in[0,T]}\mathbb{E}R_{s\wedge\zeta}^{1+1/(p-1)}\le\exp\left[\frac{p\rho_0^2(x,y)}{4(p-1)^2\theta(2-\theta)\int_0^T e^{2\int_0^s K(r)\mathrm{d}r}\mathrm{d}s}\right]$$

The proof is similar as in the proof of [3, Theorem 2], we omit it here.

2t2 **Theorem 6.3.** *Assume (4.4) holds, then*

(1) $P_{S,T}\log f(y)\le\log P_{S,T}f(x)+\frac{\rho_S^2(x,y)}{4\int_S^T e^{\int_S^r K(u)\mathrm{d}u}\mathrm{d}r}$, *for* $0\le S\le T<T_c$.

(2) *For $p>1$ and $0\le S\le T<T_c$, the Harnack inequality*

$$(P_{S,T}f(y))^p\le(P_{S,T}f^p(x))\exp\left[\frac{p\rho_S^2(x,y)}{4(p-1)\int_S^T e^{2\int_S^r K(u)\mathrm{d}u}\mathrm{d}r}\right]$$

holds $x,y\in M$ and $f\in\mathscr{B}_b^+(M)$.

Proof. (1) We only consider $S=0$ for simplicity. Since we have $\int_0^T \xi_t^{-1}\mathrm{d}t=\infty$, we get $X_T=Y_T$ $\mathbb{Q}$-a.s.. Now combining Lemma 6.1 with $X_T=Y_T$, and using the Young inequality, for $f\geq 1$ we have

$$\begin{aligned}P_{0,T}\log f(y)&=\mathbb{E}_{\mathbb{Q}}[\log f(Y_T)]=\mathbb{E}[R_{T\wedge\zeta}\log f(X_T)]\leq \mathbb{E}R_{T\wedge\zeta}\log R_{T\wedge\zeta}+\log\mathbb{E}f(X_T)\\&\leq \log P_{0,T}f(x)+\frac{\rho_0^2(x,y)}{4\theta(2-\theta)\int_0^T e^{2\int_0^r K(u)\mathrm{d}u}\mathrm{d}r}.\end{aligned}$$

This completes the proof of (1) by taking $\theta=1$.

(2) Since $X_T=Y_T$ and $\{\tilde{B}_t\}_{t\in[0,T_c)}$ is the d-dimensional Brownian motion under $\mathbb{Q}$, we have

$$(P_{0,T}f(y))^p=(\mathbb{E}_{\mathbb{Q}}[f(Y_T)])^p=(\mathbb{E}[R_{T\wedge\zeta}f(X_T)])^p\leq (P_{0,T}f^p(x))(\mathbb{E}R_{T\wedge\zeta}^{p/(p-1)})^{p-1}. \tag{6.7}$$

We complete the proof by combining Lemma 6.2. $\square$

6.2 Extension to the case with a diffusion coefficient

In this subsection, we consider the diffusion process generated by $L_t:=\psi_t^2(\Delta_t+Z_t)$ on Riemannian manifold (M,g_t) with convex boundary ∂M for all $t\in[0,T_c)$, where $\psi_t\in C^1(M)$ and Z is a $C^{1,1}$-vector field on M. Assume that ψ_t is bounded and

$$\mathrm{Ric}_t^Z\geq -K_1(t),\quad \partial_t g_t\leq K_2(t),\quad \mathbb{I}_t\geq 0 \tag{6.8}$$

hold for some continuous function on $[0,T_c)$. Then the (reflecting) diffusion process generated by L_t is non-explosive by [12, Corollary 2.2].

Let $P_{s,t}$ be the (Neumann) semigroup generated by L_t. Then $P_{s,t}$ is the semigroup for the solution to the SDE

$$\mathrm{d}_I X_t=\sqrt{2}\psi_t(X_t)u_t\mathrm{d}B_t+\psi_t^2(X_t)Z_t(X_t)\mathrm{d}t+N_t(X_t)\mathrm{d}l_t, \tag{6.9}$$

where B_t is the d-dimensional Brownian motion on probability space $(\Omega,\{\mathscr{F}_t\}_{t\geq 0},\mathbb{P})$, u_t is the horizontal lift of X_t onto the frame bundle $\mathcal{O}_t(M)$, and l_t is the local time of X_t on ∂M, when $\partial M=\varnothing$, we simply set $l_t=0$.

To derive the Harnack inequality, we assume that for $T\in(0,T_c)$,

$$\lambda_T:=\inf_{(t,x)\in[0,T]\times M}|\psi_t(x)|>0,\quad \delta_T:=\sup_{t\in[0,T]}(\sup\psi_t-\inf\psi_t)<\infty.$$

Now, let $x,y\in M$ and $T\in(0,T_c)$ be fixed. Let ρ_t be the Riemannian distance on M, i.e. $\rho_t(x,y)$ is the length of the g_t-minimal geodesic on M linking x and y, which exists if $(\partial M,g_t)$ is either convex or empty.

Let X_t solve (6.9) with $X_0=x$. Next, for strictly positive function $\xi_t\in([0,T))$, let Y_t solve

$$\mathrm{d}_I Y_t=\sqrt{2}\psi_t(Y_t)P_{X_t,Y_t}^t u_t\mathrm{d}B_t+\psi_t(Y_t)^2Z_t(Y_t)\mathrm{d}t-\frac{\psi_t(Y_t)\rho_t(X_t,Y_t)}{\psi_t(X_t)\xi_t}\nabla^t\rho_t(X_t,\cdot)(Y_t)\mathrm{d}t+N_t(Y_t)\mathrm{d}\tilde{l}_t \tag{6.10}$$

for $Y_0=y$, where $\tilde{l}_t$ is the local time of Y_t on ∂M. In the spirit of Theorem 4.1, we may assume that the cut-locus of M is empty such that the parallel displacement is smooth.

Let

$$\mathrm{d}\tilde{B}_t = \mathrm{d}B_t + \frac{\rho_t(X_t, Y_t)}{\sqrt{2}\xi_t\psi_t(X_t)} u_t^{-1}\nabla^t \rho_t(\cdot, Y_t)(X_t)\mathrm{d}t, \quad t < T. \tag{6.11}$$

By the Girsanov theorem, for any $s \in (0, T)$, the process $\{\tilde{B}_t\}_{t\in[0,s]}$ is the d-dimensional Brownian motion under the changed probability measure $R_s\mathbb{P}$, where

$$R_s := \exp\left[-\int_0^s \frac{\rho_t(X_t, Y_t)}{\sqrt{2}\xi_t\psi_t(X_t)} \left\langle \nabla^t\rho_t(\cdot, Y_t)(X_t), u_t\mathrm{d}B_t\right\rangle - \frac{1}{2}\int_0^s \frac{\rho_t(X_t, Y_t)^2}{\sqrt{2}\xi_t^2\psi_t(X_t)^2}\mathrm{d}t\right].$$

Thus, by (6.11) we have

$$\begin{aligned}
\mathrm{d}_I X_t &= \sqrt{2}\psi_t(X_t)u_t\mathrm{d}\tilde{B}_t + \psi_t^2(X_t)Z_t(X_t)\mathrm{d}t - \frac{\rho_t(X_t, Y_t)}{\xi_t}\nabla^t\rho_t(\cdot, Y_t)(X_t)\mathrm{d}t + N_t(X_t)\mathrm{d}l_t;\\
\mathrm{d}_I Y_t &= \sqrt{2}\psi_t(Y_t)P^t_{X_t,Y_t}u_t\mathrm{d}\tilde{B}_t + \psi_t^2(Y_t)Z_t(Y_t)\mathrm{d}t + N_t(Y_t)\mathrm{d}\tilde{l}_t.
\end{aligned}$$

By the second variational formula and the index lemma,

$$\begin{aligned}
\mathrm{d}\rho_t(X_t, Y_t) \le& \sqrt{2}(\psi_t(X_t) - \psi_t(Y_t))\left\langle \nabla^t\rho_t(\cdot, Y_t)(X_t), u_t\mathrm{d}\tilde{B}_t\right\rangle_t - \frac{\rho_t(X_t, Y_t)}{\xi_t}\mathrm{d}t\\
&+ \left\{\sum_{i=1}^d U_i^2(t, X_t, Y_t)\rho_t(X_t, Y_t) + \psi_t^2(X_t)\left\langle \nabla^t\rho_t(\cdot, Y_t)(X_t), Z_t(X_t)\right\rangle_t\right.\\
&\left. + \psi_t^2(Y_t)\left\langle \nabla^t\rho_t(X_t, \cdot)(Y_t), Z_t(Y_t)\right\rangle_t + \frac{1}{2}\int_0^{\rho_t} \partial_t g_t(\dot{\gamma}(s), \dot{\gamma}(s))\mathrm{d}s\right\}\mathrm{d}t
\end{aligned}$$

where $\{U_i\}_{i=1}^{d-1}$ are vector fields on $M \times M$ such that $\nabla^t U_i(t, X_t, Y_t) = 0$ and

$$U_i(t, X_t, Y_t) = \psi_t(X_t)V_i + \psi_t(Y_t)P^t_{X_t,Y_t}V_i, \quad 1 \le i \le d-1$$

for $\{V_i\}_{i=1}^d$ an g_t-OBN of $T_{X_t}M$ with $V_d = \nabla^t\rho_t(\cdot, Y_t)(X_t)$. ρ_t is short for $\rho_t(X_t, Y_t)$ and γ is the g_t-geodesic connecting X_t and Y_t. With a similar discussion as in Proposition 4.3, we obtain the following estimation.

$$\begin{aligned}
\mathrm{d}\rho_t(X_t, Y_t) \le& \sqrt{2}(\psi_t(X_t) - \psi_t(Y_t))\left\langle \nabla^t\rho_t(\cdot, Y_t)(X_t), u_t\mathrm{d}\tilde{B}_t\right\rangle_t + K_\psi(t)\rho_t(X_t, Y_t)\mathrm{d}t\\
&- \frac{\rho_t(X_t, Y_t)}{\xi_t}\mathrm{d}t, \quad t < T,
\end{aligned} \tag{6.12}$$

where $K_\psi(t) = (d-1)\|\nabla^t\psi_t\|_\infty^2 + K_1(t)\|\psi_t\|_\infty^2 + 2\|Z_t\|_\infty\|\psi_t\|_\infty\|\nabla^t\psi_t\|_\infty + \frac{1}{2}K_2(t)$. This implies that

$$\begin{aligned}
\mathrm{d}\frac{\rho_t(X_t, Y_t)^2}{\xi_t} \le& \frac{2\sqrt{2}}{\xi_t}\rho_t(X_t, Y_t)(\psi_t(X_t) - \psi_t(Y_t))\left\langle \nabla^t\rho_t(\cdot, Y_t)(X_t), u_t\mathrm{d}\tilde{B}_t\right\rangle_t\\
&- \frac{\rho_t(X_t, Y_t)^2}{\xi_t^2}\left[\xi_t' - (2\|\nabla^t\psi_t\|_\infty^2 + 2K_\psi(t))\xi_t + 2\right]\mathrm{d}t
\end{aligned}$$

Let

$$\widehat{K}(t) := 2K_\psi(t) + 2\|\nabla^t\psi_t\|_\infty^2 = 2d\|\nabla^t\psi_t\|_\infty^2 + 2K_1(t)\|\psi_t\|_\infty^2 + 4\|Z_t\|_\infty\|\psi_t\|_\infty\|\nabla^t\psi_t\|_\infty + K_2(t).$$

In particular, letting

$$\xi_t = (2-\theta)e^{2\int_0^t \widehat{K}(r)\mathrm{d}r}\left(\int_0^T e^{-2\int_0^s \widehat{K}(r)\mathrm{d}r}\mathrm{d}s - \int_0^t e^{-2\int_0^s \widehat{K}(r)\mathrm{d}r}\mathrm{d}s\right), \quad t \in [0, T_c),\ \theta \in (0, 2).$$

We have

$$2 - \widehat{K}(t)\xi_t + \xi_t' = \theta.$$

Therefore, the following results following immediately by combining the discussion as the case with constant coefficient.

Theorem 6.4. *Assume (6.8) holds. Let Z_t, ψ_t be bounded on M such that*

$$\widehat{K}(t) := 2K_1(t)\|\psi_t\|_\infty^2 + 4\|Z_t\|_\infty\|\psi_t\|_\infty\|\nabla^t\psi_t\|_\infty + 2d\|\nabla^t\psi_t\|_\infty^2 + K_2(t) < \infty$$

for $t \in [0, T_c)$. Then

(1) $P_{S,T}\log f(y) \le \log P_{S,T}f(x) + \frac{\rho_S^2(x,y)}{4\lambda_T^2\int_S^T e^{-2\int_S^u \widehat{K}(r)\mathrm{d}r}\mathrm{d}s}$, $f \ge 1$, $x, y \in M$, $0 \le S < T < T_c$.

(2) *For $p > (1+\frac{\delta_T}{\lambda_T})^2$, $\delta_p := \max\{\delta_T, \frac{\lambda_T}{2}(\sqrt{p}-1)\}$ and $0 \le S < T < T_c$, the Harnack inequality*

$$(P_{S,T}f(y))^p \le (P_{S,T}f^p(x))\exp\left[\frac{\sqrt{p}(\sqrt{p}-1)\rho_S(x,y)}{8\delta_p[(\sqrt{p}-1)\lambda_T - \delta_p]\int_S^T e^{-2\int_S^r \widehat{K}(u)\mathrm{d}u}\mathrm{d}r}\right].$$

6.3 Extension to manifolds with convex boundary

In this subsection, we come back to consider the process X_t generated by $L_t = \Delta_t + Z_t$ on the time-inhomogeneous space with boundary, which is possible non-convex under some g_t. Assume that $\mathscr{D} \ne \varnothing$ and for some $K_1, K_2 \in C([0, T_c))$ such that

$$\mathrm{Ric}_t^Z \ge -K_1(t), \quad \partial_t g_t \le K_2(t) \tag{6.13}$$

holds. Let $\phi_t \in \mathscr{D}$, then as explained, $\tilde{g}_t = \phi_t^{-2}g_t$ is a convex flow. Since $\phi_t \ge 1$, $\rho_t(x,y)$ is large than $\tilde{\rho}_t(x,y)$, the Riemannian $\tilde{g}_t$-distance between x and y. Recall the results induced from the proof of Proposition 4.3,

$$L_t = \phi_t^{-2}(\tilde{\Delta}^t + \tilde{Z}_t), \quad \widetilde{\mathrm{Ric}}_t^{\tilde{Z}} \ge -K_{\phi,1}(t), \quad \partial_t\tilde{g}_t \le K_{\phi,2}(t),$$

where $\tilde{Z}_t = \phi_t^2 Z_t + \frac{d-2}{2}\nabla^t\phi_t^2$,

$$K_{\phi,1}(t) = \sup_M\{K_1(t)\phi_t^2 - \phi_t\Delta_t\phi_t + (d-3)|\nabla^t\phi_t|_t^2 + 2|Z_t|_t\phi_t|\nabla^t\phi_t|_t\},$$

and

$$K_{\phi,2}(t) = \sup_M\{K_2(t) - 2\phi_t^{-1}\partial_t\phi_t\}.$$

Applying Theorem 6.4 to the convex manifold $(M, \tilde{g}_t)$, $t \in [0, T_c)$, $\psi_t = \phi_t^{-1}$ and

$$K_\phi(t) = 2K_{\phi,1}(t) + 4\|\phi_t Z_t + (d-2)\nabla^t\phi_t\|_\infty\|\nabla^t\phi_t\|_\infty + 2d\|\nabla^t\phi_t\|_\infty^2 + K_{\phi,2}(t) < \infty. \tag{6.14}$$

We obtain the following result.

Theorem 6.5. *Assume (6.13) holds for $K_1, K_2 \in C([0, T_c))$. For any $\phi \in \mathscr{D}$, let $K_\phi(t)$ be fixed by (6.14). Then all assertion in Theorem 6.4 holds for continuous function K_ϕ instead of $\hat{K}$, $\delta_T := 1 - \sup_{t\in[0,T]}\inf\phi_t^{-1}$, and $\lambda_T := \inf_{[0,T]\times M}\phi_t^{-1}$.*

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